WILSON LOOPS AND FREE ENERGIES IN $3d$ $\mathcal{N}=4$ SYM: EXACT RESULTS, EXPONENTIAL ASYMPTOTICS AND DUALITY

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Abstract. We show that $U(N)$ $3d$ $\mathcal{N}=4$ supersymmetric gauge theories on S^3 with N_f massive fundamental hypermultiplets and with a Fayet-Iliopoulos (FI) term are solvable in terms of generalized Selberg integrals. Finite N expressions for the partition function and Wilson loop in arbitrary representations are given. We obtain explicit analytical expressions for Wilson loops with symmetric, antisymmetric, rectangular and hook representations, in terms of Gamma functions of complex argument. The free energy for orthogonal and symplectic gauge group is also given. The study of asymptotic expansions of the free energy then leads to the emergence of exponentially small contributions for $N_f < 2N-2$, which corresponds to *bad* theories. Duality checks are also explicitly performed and we show how the exponential asymptotics is understood from the point of view of the duality between *good* and *bad* theories

1. Introduction

The study of supersymmetric gauge theories in compact manifolds has been considerably pushed forward in recent years, after the development of the localization method [1], which reduces the original functional integral describing the quantum field theory into a much simpler finite-dimensional integral. The task of computing observables in a supersymmetric gauge theory then typically consists in analyzing a resulting integral representation, which is normally of the matrix model type. There is a large number of tools available in the study of matrix models. We will use here results from random matrix theory and the theory of the Selberg integral [2].

We study in particular a three-dimensional $\mathcal{N}=4$ gauge theory on S^3, which consists of a $U(N)$ vector multiplet coupled to N_f massive hypermultiplets in the fundamental representation, together with a Fayet-Iliopoulos (FI) term. The rules of localization in three dimensions [3, 4], immediately gives the corresponding matrix model expression for the partition function

$$Z_{N_f}^{U(N)} = \frac{1}{N!}\int d^N\mu \frac{e^{\mathrm{i}\eta\sum_i \mu_i}}{\prod_i \left(2\cosh(\frac{1}{2}(\mu_i+m))\right)^{N_f}} \prod_{i<j} 4\sinh^2(\frac{1}{2}(\mu_i-\mu_j)), \tag{1.1}$$

A more general case, also including adjoint hypermultiplets has also been studied, for example in [5, 6, 7]. In the case of one adjoint hypermultiplet, the theory is known to be dual to M-theory on $AdS_4 \times S^7/N_f$, where the quotient by N_f leads to an A_{N_f-1} singularity. While our results could be extended to this more complicated model, we focus here on (1.1). In what follows, we do not include the customary $(1/N!)$ factor in (1.1) and simply call the partition function Z_N. We will comment on that term at the end of the paper, when discussing dualities.

Other recent exact analytical evaluations of free energies of $3d$ supersymmetric gauge theories can be found in [8, 9]. The models studied there are more general, but due to remarkable identities satisfied by the double sine functions that appear in their matrix models, can also be analyzed analytically. However, we still find it valuable to focus on the simpler (1.1), relate it to an exact solvable model, and extend the study to Wilson loops. In turn, this leads to non-trivial aspects of the role of the FI parameter term and an ensuing discussion of exponentially small contributions in asymptotic expansions, in relationship with the classification of these 3d theories as *good*, *ugly* or *bad* [10, 11, 12].

In addition, notice that, in spite of the apparent simplicity of this model, even simpler models in 3d, like the Abelian gauge theory studied in [13], exhibit rich behavior such as large N_f phase transitions. This non-triviality of the models is in large part due to the presence of the FI parameter, which always implies an oscillatory Fourier kernel in the matrix model representation. Concerning (1.1), we will give explicit expressions for the free energy at finite and large N, and likewise for the average of a Wilson loop in arbitrary representation, which is given by the matrix integral

$$\langle W_\lambda(N)\rangle = \frac{1}{Z}\int d^N\mu \frac{s_\lambda(e^{\mu_1},...,e^{\mu_N})\ e^{\mathrm{i}\eta\sum_i \mu_i}}{\prod_i \left(2\cosh(\frac{1}{2}(\mu_i+m))\right)^{N_f}} \prod_{i<j} 4\sinh^2(\frac{1}{2}(\mu_i-\mu_j)), \tag{1.2}$$

where Z refers to (1.1) and s_λ denotes a Schur polynomial. Aspects of the analysis of Wilson loops in this theory, in particular the mirror symmetry between Wilson loops and vortex loops, have recently appeared in [7]. The matrix models have been considered without FI term in [5, 6] and with one in [7].

The case where all the N_f masses are different can also be solved exactly for the partition function [8] and has been studied and used in a number of works [14, 15]. The solution in that case is based on a Cauchy determinant formula. Thus, the limiting case, when all masses become equal, is not immediate and it seems that even the partition function of the model has not been analyzed as the confluent limit of the expression when all masses are distinct. We show below how the large N behavior of the free energy has a different leading behavior when all masses are equal, since the usual leading term $\left(N^2/2\right)\log N$ [14, 15, 16] cancels out. Other, in principle more general versions of the matrix model, dealing for example with the case where the matter content consists of $\mathcal{N}=2$ mass deformation of $\mathcal{N}=3$ hypermultiplets, have been studied in [17], using Cauchy theorem and in the context of factorization of 3d partition functions [18]. In this work, infinite series expressions are obtained and hence asymptotics could not be obtained with such formulas. In contrast to the previous works, our explicit analytical expressions will all be in terms of special functions, G-Barnes functions and Gamma functions, of complex argument.

Notice that in the Abelian case, the theory is exactly solvable, since the observables then reduce to the evaluation of a very well-known Fourier transform

$$Z_t\left(\eta\right) = \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}\eta\mu}d\mu}{\left(\cosh(\mu+m)\right)^t} = \frac{\Gamma\left(\left(t+i\eta\right)/2\right)\Gamma\left(\left(t-i\eta\right)/2\right)}{\Gamma\left(t\right)}, \tag{1.3}$$

which follows by immediate identification with Euler's beta integral

$$\int_{-\infty}^{\infty}\frac{ce^{-qcy}dy}{(1+e^{-cy})^{p+q}} = \int_0^1 t^{p-1}(1-t)^{q-1}dt := B(p,q) = \frac{\Gamma\left(p\right)\Gamma\left(q\right)}{\Gamma\left(p+q\right)}, \tag{1.4}$$

with $\Re(c)>0$, $\Re(p)$, $\Re(q)\geq 0$ and $\Gamma\left(p\right)$ the Gamma function.

The following exact evaluations, given in terms of a number of Barnes G-functions, admit the analysis of the observables for different large values of the parameters, such as N, N_f and η, leading to asymptotic series. The presence of the FI parameter makes the analysis richer, since it then involves complex arguments z for the Barnes G-functions and the Gamma functions below.

Therefore, the large z behavior of the special functions is required, involving naturally the whole complex plane, which leads to consideration of Stokes and anti-Stokes lines and the appearance of exponentially small contributions when crossing them. We can analyze the observables in these terms by using the exponential asymptotics of Barnes and Gamma functions [19, 20, 21]. Therefore, this theory is suitable to further explore ideas of resurgence and resummation, which have become a subject of considerable interest in the study of gauge theories in recent years (see

[21, 22], for example, [23, 24, 25] for supersymmetric gauge theories and localization and [26] for a review).

It is important to stress however, that the issue of Stokes lines crossing will not appear here while moving the physical parameters (which is essentially here the FI parameter) of a given theory but rather, we will see that the crossing occurs will occur when the number of flavours is smaller than a certain value, in which case the theory becomes a *bad* theory.

The paper is organized as follows. In the next Section, we show how (1.1) and (1.2) can be evaluated by mapping the model into a random matrix ensemble and using the results in [27], which contains, among other results, a new extension of the Selberg integral. We show that the analytical continuation of the Beta function (1.4) extends the results to any number of flavours and to a non-zero FI parameter η. We use this also to obtain an analytical expression for the Wilson loop. Four specific sets of Wilson loops with FI parameter are studied in detail: symmetric, antisymmetric, rectangular and hook representations, obtaining explicit expressions in terms of Gamma functions of complex argument.

In Section 3, we focus exclusively on the partition function and show that the matrix model (1.1) and its counterparts for the orthogonal and symplectic gauge groups, are all easily mapped into the Selberg integral, which gives an exact evaluation of the free energies in terms of G-Barnes functions. We also study the asymptotics for $\eta = 0$ of the $U(N)$ free energy for large N and constant Veneziano parameter $\zeta = N_f/N$, showing that the leading term is $f(\zeta) N^2$ instead of $N^2 \log N$ and we determine $f(\zeta)$ and the subleading contributions to the free energy. An integral representation of the Mellin-Barnes type is given for the case of a $SU(N)$ gauge group and the $SU(2)$ case is computed explicitly in two ways.

In the last Section we study the asymptotics mainly of the free energy, with FI parameter, discussing the crossing of Stokes lines and the ensuing appearance of exponentially small contributions in the asymptotic expansion of the partition functions of *bad* theories and its absence for *ugly* and *good* theories. Explicit duality tests are carried out for the partition function and the decoupled free sectors characterized by Gamma functions are shown to exactly account for the previously observed difference in the asymptotics, with regards to exponentially small contributions, of *good* and *bad* theories. We conclude with some open directions for further work.

2. Exact evaluation of the Wilson loop average

We want the analytical evaluation (1.2). For this, notice that the usual change of variables $e^{\mu} = y$,[1] useful when matrix model contains the hyperbolic version of the Vandermonde determinant, immediately leads to the following matrix model

$$\begin{aligned} \langle W_{\lambda}(N) \rangle &= \frac{1}{Z} \int_{[0,\infty)^N} d^N y \prod_{i=1}^{N} \frac{e^{\frac{mN_f}{2}} y_i^{i\eta + \frac{N_f}{2} - N}}{(1+e^m y_i)^{N_f}} s_{\lambda}(y_1, ..., y_N) \prod_{i<j} (y_i - y_j)^2 \\ &= \frac{e^{-mN\left(i\eta + \frac{|\lambda|}{N}\right)}}{Z} \int_{[0,\infty)^N} d^N x \prod_{i=1}^{N} \frac{x_i^{i\eta + \frac{N_f}{2} - N}}{(1+x_i)^{N_f}} s_{\lambda}(x_1, ..., x_N) \prod_{i<j} (x_i - x_j)^2. \end{aligned} \tag{2.1}$$

The mass dependence is accounted for in the prefactor

$$\exp\left[-mN\left(i\eta + \frac{|\lambda|}{N}\right)\right],$$

[1]Precisely, we use $e^{\mu} = y$ and then $ye^m = x$.

and absent in the matrix model itself, as can be also simply seen directly from (1.1). The same holds for the normalizing partition function term Z, since it has the same matrix model representation (2.1) but without the Schur polynomial. Therefore, cancelling the common mass-dependent prefactor, overall, only the term $\exp\left(-m\left|\lambda\right|\right)$ will remain.

In contrast to the case of Chern-Simons theory, the absence of a Gaussian factor in the matrix model representation (1.1) implies that the $x^{i\eta+N_f/2-N}$ factor in the weight function is not removed with a shift of the above change of variables, such as in [28]. Without such term the integrand in (2.1) is known to represent the joint probability distribution function of an ensemble of complex matrices [29]. The model can then be solved by using for example[2] $\prod_{i=1}^{N} x_i^n s_\lambda(x_1,...,x_N) = s_{\lambda+(n^N)}(x_1,...,x_N)$, together with the exact results in [27, Lema 3, (a)]. However, directly employing the later work [30] instead, gives an immediate solution.

The implication of this solution for the gauge theories is what we discuss here. The central result in [30], for us, is the following: Let k and n be nonnegative integers[3], then for any partition λ such that $l(\lambda) \leq k$ and $l(\lambda') \leq n$

$$\frac{s_\lambda(1_n)s_\lambda(1_k)}{s_{\lambda'}(1_a)} = \frac{1}{C^a_{n,k}} \int_0^\infty \ldots \int_0^\infty s_\lambda(t_1,\ldots t_k)\Delta^2(t_1,\ldots,t_k) \prod_{j=1}^{k} \frac{t_j^{n-k} dt_j}{(1+t_j)^{a+n+k}}, \tag{2.2}$$

with

$$C^a_{n,k} = \int_0^\infty \ldots \int_0^\infty \Delta^2(t_1,\ldots,t_k) \prod_{j=1}^{k} \frac{t_j^{n-k} dt_j}{(1+t_j)^{a+n+k}} = \prod_{j=0}^{k-1} \frac{\Gamma\left(j\right)\Gamma(j+n-k+1)\Gamma(a+j+1)}{\Gamma(a+n+j+1)}. \tag{2.3}$$

The later expression follows from Selberg's integral [27, 30, Lema 3, (a)] whereas (2.2) is a novel extension of the Selberg integral, obtained in [30] and previously, in a simpler form, in [27, Lema 3, (a)]. Notice that the double constraint above on the partition λ of the Schur polynomial, necessarily implies that[4] $N_f > 2N$ which is also the regime identified in [7]. Indeed, for these values the integrals are manifestly convergent. This regime corresponds to the *good* or *ugly* classification in [10]. In the last Section, while studying the asymptotics of the observables, we will be naturally lead also into considering the setting $N < N_f < 2N$, which corresponds to *bad* theories.

As we see from the expressions below, involving Schur polynomials, in principle it seems we need to consider the case of an even number of flavours and take, at least for the moment the restricted view, above exposed, for the FI parameter. For definiteness, we take now $\eta = 0$ but then below, we show how to lift this restriction, using the analytical continuation given by the beta function (1.4). The Wilson loop is then

$$\langle W_\lambda(N)\rangle = e^{-m|\lambda|} \frac{s_\lambda(1_{N_f/2})s_\lambda(1_N)}{s_{\lambda'}(1_{N_f/2-N})}, \tag{2.4}$$

which, by using the Weyl dimension formula for the specialization of the Schur polynomials

$$s_\mu(1^N) = \frac{1}{G(N+1)} \prod_{1\leq j<k\leq N} (\mu_j - \mu_k + k - j), \tag{2.5}$$

[2]This leads to a discussion of the FI term, as this argument seemingly requires that the FI is either taken to be 0 (as is the case in [6]), or it is such that $i\eta \in \mathbb{Z}$. The latter choice has been discussed in order to avoid restriction on the charge (size of the partition) of the Wilson loop average [7] (where the choice is implemented through a modification of the contour integration).

[3]We slightly change their notation, since their m parameter could be confussed with the mass here.

[4]Recall that $s_\lambda(t_1,\ldots t_m) = 0$ if $l(\lambda) > m$.

valid when $l(\mu) \leq N$, can also be written as
(2.6)

$$\langle W_\lambda(N)\rangle = \frac{e^{-m|\lambda|}G(N_F/2-N+1)}{G(N+1)G(N_F/2+1)}\frac{\prod_{1\leq j<k\leq N}(\lambda_j-\lambda_k+k-j)\prod_{1\leq j<k\leq N_F/2}(\lambda_j-\lambda_k+k-j)}{\prod_{1\leq j<k\leq_{N_F/2-N}}(\lambda_j'-\lambda_k'+k-j)}$$

and the partition function, from (2.3) is

$$Z_N = e^{-imN\eta}\frac{G\left(N+2\right)G(\frac{N_f}{2}-i\eta+1)G(\frac{N_f}{2}+i\eta+1)G(N_f-N+1)}{G(\frac{N_f}{2}-N-i\eta+1)G(\frac{N_f}{2}-N+i\eta+1)G(N_f+1)}. \tag{2.7}$$

Looking for potential poles or zeroes of the expressions, recall that the G-Barnes function is an entire function and its zeroes are located at $G(-n)=0$ for $n=0$ and $n\in\mathbb{N}$. As expected, there is a drastic difference with regards to convergence, according to the FI parameter. For $\eta=0$, the two G-Barnes in the denominator do give zeroes, precisely for $N_f<2N-1$ and we have the known, expected, divergence of the partition function in this case. This is the well-known divergence of the partition function [12].

For $\eta\neq 0$, there is no divergence of the partition function, since the G-Barnes of the denominators do not have a zero there, for a bad theory with $N_f\leq 2(N-1)$. The case of complex η [31] will be briefly touched upon in the last Section. Notice that the G-Barnes function part in (2.7) is symmetric under $\eta\to-\eta$. Therefore, due to the imaginary prefactor $Z_N(-\eta)=\overline{Z_N(\eta)}$, as also happens when there is a Chern-Simons term [31, 32]. Note also that (2.7) admits alternative equivalent expressions, for example involving Gamma functions. This will be relevant below when discussing the $SU(N)$ case and also at the end, when studying duality.

Regarding Wilson loops it seems that the required specialization of Schur polynomials puts some restriction on the parameters of our model but actually these admit an expression in trms of Beta functions and this provides an extension to the whole complex plane. We show this explicitly now.

2.1. **Symmetric and antisymmetric representations.** Thus, we focus now on two interesting specific instances of (2.4), namely antisymmetric representation and symmetric representation. Recall how difficult these specific cases are to analyze in $\mathcal{N}=4$ $U(N)$ SYM theory in four dimensions, even though the corresponding matrix model there is a Gaussian ensemble (see [33] for a recent review).

In contrast, in this case, the solvability of the model, which is in general that of a multidimensional beta function (Selberg integral), reduces to that of the ordinary beta integral. This leads to compact exact expressions valid for all N. Precisely, recalling the Euler integral identity (1.4), then the elementary and homogeneous symmetric polynomials are given in terms of the Beta function

$$\frac{1}{e_r(1_n)} = (n+1)\int_0^\infty \frac{t^r}{(1+t)^{n+2}}dt \quad \text{and} \quad \frac{1}{h_r(1_n)} = (n-1)\int_0^1 t^r(1-t)^{n-2}dt. \tag{2.8}$$

Then, with λ a partition of length 1, $\lambda=(r)$, we have that

$$s_\lambda(1_n) = h_r(1_n) = \binom{n+r-1}{r} \quad \text{and} \quad s_{\lambda'}(1_n) = e_r(1_n) = \binom{n}{r}. \tag{2.9}$$

The general Wilson loop expression (2.4) adopts a very concrete expression in terms of Gamma functions, for a symmetric representation

$$\langle W_{(r)}(N)\rangle = e^{-mNr}\frac{\Gamma\left(\frac{N_f}{2}+r\right)\Gamma\left(N+r\right)\Gamma\left(\frac{N_f}{2}-N-r+1\right)}{\Gamma\left(r+1\right)\Gamma\left(\frac{N_f}{2}\right)\Gamma\left(N\right)\Gamma\left(\frac{N_f}{2}-N+1\right)},$$

and the antisymmetric representation

$$\left\langle W_{(1)^r}(N)\right\rangle = \frac{e^{-mNr}\Gamma\left(\frac{N_f}{2}+1\right)\Gamma\left(N+1\right)\Gamma\left(\frac{N_f}{2}-N\right)}{\Gamma\left(r+1\right)\Gamma\left(\frac{N_f}{2}-r+1\right)\Gamma(N-r+1)\Gamma(\frac{N_f}{2}-N+r)}.$$

Notice that these expressions do not restrict N_f to be even. Likewise, the expressions in terms of Gamma functions suggest the consideration of the FI parameter. In that case, we would have

$$\langle W_\lambda(N,\eta)\rangle = e^{-m|\lambda|}\frac{s_\lambda(\mathbf{1}_{\frac{N_f}{2}+i\eta})s_\lambda(\mathbf{1}_N)}{s_{\lambda'}(\mathbf{1}_{\frac{N_f}{2}-N-i\eta})}. \tag{2.10}$$

Because of (2.8), each term in this expression is a Beta function, two of these have one of their two parameters complex, so one can use (1.4)[5] and the resulting expressions with a generic N_f and a non-zero FI parameter are then given by

$$\left\langle W_{(r)}(N,\eta)\right\rangle = e^{-mr}\frac{\Gamma\left(i\eta+\frac{N_f}{2}+r\right)\Gamma\left(N+r\right)\Gamma\left(\frac{N_f}{2}-N-i\eta-r+1\right)}{\Gamma\left(r+1\right)\Gamma\left(\frac{N_f}{2}+i\eta\right)\Gamma\left(N\right)\Gamma\left(\frac{N_f}{2}-N-i\eta+1\right)}, \tag{2.11}$$

$$\left\langle W_{(1)^r}(N,\eta)\right\rangle = \frac{e^{-mr}\Gamma\left(\frac{N_f}{2}+i\eta+1\right)\Gamma\left(N+1\right)\Gamma\left(\frac{N_f}{2}-N-i\eta\right)}{\Gamma\left(r+1\right)\Gamma\left(\frac{N_f}{2}-r+i\eta+1\right)\Gamma(N-r+1)\Gamma(\frac{N_f}{2}-N-i\eta+r)}. \tag{2.12}$$

Note that in these expressions the Gamma functions can be traded for Pochhammer symbols, using $r-1$ times $\Gamma(z+1)=z\Gamma(z)$. Some simple checks can be quickly made. For example for $r=0$ the expressions above give 1, as expected and for $r=1$ (fundamental representation) both expressions coincide, giving

$$\left\langle W_{(1)}(N,\eta)\right\rangle = e^{-m}\frac{N\Gamma\left(i\eta+\frac{N_f}{2}+1\right)\Gamma\left(\frac{N_f}{2}-N-i\eta\right)}{\Gamma\left(\frac{N_f}{2}+i\eta\right)\Gamma\left(\frac{N_f}{2}-N-i\eta+1\right)} = Ne^{-m}\frac{i\eta+\frac{N_f}{2}}{\frac{N_f}{2}-N-i\eta}.$$

Likewise, as for the partition function (but this time due to the Gamma functions and not the exponential prefactor, which is real), we have that $\langle W_\mu(N,-\eta)\rangle = \overline{\langle W_\mu(N,\eta)\rangle}$.

The case of antisymmetric representation with N boxes can be seen as a partition function computation because $s_{(1^N)}(x_1,...,x_N) = e_N(x_1,...,x_N) = \prod_{i=1}^N x_i$, see below for this point of view in the more general case of a rectangular representation. Here, using (2.12) we have that

$$\left\langle W_{(1^N)}(N,\eta)\right\rangle = e^{-mN}\frac{\Gamma\left(\frac{N_f}{2}+i\eta+1\right)\Gamma\left(\frac{N_f}{2}-N-i\eta\right)}{\Gamma\left(\frac{N_f}{2}-N+i\eta+1\right)\Gamma\left(\frac{N_f}{2}-i\eta\right)}. \tag{2.13}$$

Notice that it is as the result for the fundamental but with the complex conjugate in the denominator.

[5]A way to prove (1.4) is by induction when p is an integer, and since the integral is bounded and analytical for $\Re(p), \Re(q) \geq 0$ -and so is the r.h.s. of the formula- then by Carlson theorem [34], the expression follows for complex p and q. However, to prove uniqueness of the analytical extension of the Wilson loops, one needs to guarantee also that the factorized expression (2.10) also holds for complex values. Here, we just extended each piece in a unique way, by using the Beta function.

2.2. **Rectangular partitions.** There are more general representations that also can be studied very explicitly. In particular, the case of rectangular partitions is specially interesting. Recall the statement made above, before (2.2), on rectangular partitions: if we consider the partition of length N, $(l, l, ..., l)$ which we denote by $l^N = (l^N)$ then, assuming that λ is a partition of length equal or lower than N, we have that

$$s_{\lambda+l^N}(x_1, ..., x_N) = e_N^l(x_1, ..., x_N) s_\lambda(x_1, ..., x_N), \tag{2.14}$$

which follows by recalling that $e_N(x_1, ..., x_N) = \prod_{i=1}^N x_i$. Thus, as a simple extension of the result above for the $\left(1^N\right)$ representation, the case where the representation is described by a rectangular partition (with number of rows equal to the rank N of the gauge group), the matrix model giving the Wilson loop has the same form as the one for the partition function, with a shift of parameters. Thus,

$$\langle W_{l^N}(N, N_f, \eta)\rangle = \frac{Z_N(N, N_f, \eta - il)}{Z_N(N, N_f, \eta)} = e^{-mNl} \frac{\prod_{j=1}^{l} \Gamma\left(\frac{N_f}{2} + i\eta + 1 + l - j\right) \Gamma\left(\frac{N_f}{2} - N - i\eta + 1 - j\right)}{\prod_{j=1}^{l} \Gamma\left(\frac{N_f}{2} - i\eta + 1 - j\right) \Gamma\left(\frac{N_f}{2} - N + i\eta + l + 1 - j\right)},$$

where, in addition to taking the quotient using (2.7), we have also iteratively applied $G(z+1) = \Gamma(z)G(z)$, which also will be crucial below to check Seiberg duality. Notice that for $l = 1$, it coincides with (2.13), as it should.

Likewise, from (2.14) it follows that

$$\left\langle W_{\lambda+l^N}(N, N_f, \eta)\right\rangle = \langle W_\lambda(N, N_f, \eta - il)\rangle$$

Thus, this case is equivalent to that of a partition function with a complex FI parameter. This setting will be briefly discussed again at the end, when considering the asymptotics of G-Barnes functions and the crossing of Stokes lines.

2.3. **Hook representations.** The last explicit case that we analyze is the one corresponding to partitions represented by hooks, $\lambda = (r - s, 1^s)$. As above $|\lambda| = r$. Notice that this notation describes a Young tableaux of a row of size $r - s$ and a column of size $s + 1$ as 1^s contains s boxes below the upper-left box of the tableaux. Therefore, $\lambda' = (s + 1, 1^{r-s-1})$ and we have

$$s_\lambda(1_n) = \frac{\Gamma(n + r - s)}{r\Gamma(r - s)\Gamma(s + 1)\Gamma(n - s)} \quad \text{and } s_{\lambda'}(1_n) = \frac{\Gamma(n + s + 1)}{r\Gamma(s + 1)\Gamma(r - s)\Gamma(n - r + s + 1)}, \tag{2.15}$$

notice that for $s = 0$, then $\lambda = (r)$ and $\lambda' = (1, 1^{r-1}) = (1^r)$ and the above expressions reduce to the ones for the homogeneous and elementary symmetric polynomials, respectively (2.9) as it should. Likewise, the same consistency check is done for the dual situation, given by $s = r - 1$. Then, again using (2.10), we obtain

$$\left\langle W_{(r-s,1^s)}(N, \eta)\right\rangle = \frac{e^{-mr}\Gamma\left(\frac{N_f}{2} + i\eta + r - s\right)\Gamma(N + r - s)\Gamma(\frac{N_f}{2} - N - i\eta - r + s + 1)}{r\Gamma(r - s)\Gamma(s + 1)\Gamma(\frac{N_f}{2} + i\eta - s)\Gamma(N - s)\Gamma\left(\frac{N_f}{2} - N - i\eta + s + 1\right)}, \tag{2.16}$$

if $s = 0$ this expression indeed reduces to (2.11) and if $s = r - 1$ it then gives (2.12). Notice that, alternatively to (2.10), one could also use (2.16), together with Giambelli determinant formula, to obtain a Wilson loop in other representations, out of the hook expressions.

The asymptotics of these Wilson loop expressions is particularly rich because having complex arguments in the Gamma functions, then the crossing of Stokes lines (and related phenomena, like Berry smoothening transitions across lines [35], etc.) appears. We will discuss this at the end focussing more on the free energy.

3. $O(2N), O(2N+1)$ and $Sp(2N)$ cases, $U(N)$ asymptotics and integral representation for $SU(N)$

In this Section, we show that the partition function of the matrix models also follows, after some change of variables, from the famous evaluation of the Selberg integral [2]

$$\begin{aligned} S_N(\lambda_1,\lambda_2,\gamma) \quad : \quad &= \int_0^1 \cdots \int_0^1 \prod_{i=1}^{N} t_i^{\lambda_1}(1-t_i)^{\lambda_2} dt_i \prod_{1\leq i<j\leq n} |t_i - t_j|^{2\gamma} \\ &= \prod_{j=0}^{N-1} \frac{\Gamma(\lambda_1+1+j\gamma)\Gamma(\lambda_2+1+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\lambda_1+\lambda_2+2+(N+j-1)\gamma)\Gamma(1+\gamma)}. \end{aligned} \tag{3.1}$$

The evaluation of this integral is valid for complex parameters $\lambda_1, \lambda_2, \gamma$ such that

$$\Re(\lambda_1) > 0,\ \Re(\lambda_2) > 0,\ \Re(\gamma) > -\min\{1/N, \Re(\lambda_1)/(N-1), \Re(\lambda_2)/(N-1)\}, \tag{3.2}$$

corresponding to the domain of convergence of the integral. This evaluation is useful not only for the $U(N)$ case above discussed, but also when the gauge group is the symplectic or the orthogonal group. These latter cases were studied with the Fermi gas formalism in [5]. We conclude this Section with a discussion on how to obtain the $SU(N)$ case by integration of the $U(N)$ result (2.7) over the FI parameter.

3.1. $U(N)$ case revisited and free energy behavior for large N.

Notice that, with only a very minor modification of a change of variables proposed in [2, Ex. 4.1.3]

$$t_l = \frac{1}{e^{(s_l+m)/2}+1},$$

the Selberg integral can be written as

$$\begin{aligned} S_N(\lambda_1,\lambda_2,\lambda) \quad &= \quad e^{-(\lambda_1-\lambda_2)mN/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{e^{-(\lambda_1-\lambda_2)s_i/2} ds_i}{\left(2\cosh(\frac{1}{2}(s_i+m))\right)^{\lambda_1+\lambda_2+2+2\lambda(N-1)}} \\ &\times \prod_{i<j} \left(2\sinh(\frac{1}{2}(s_i-s_j))\right)^{2\lambda}, \end{aligned}$$

and hence, choosing $\lambda = 1$, $\lambda_1 = (2\mathrm{i}\eta - 2N + N_f)/2$ and $\lambda_2 = (N_f - 2N - 2\mathrm{i}\eta)/2$, we have that

$$\begin{aligned} Z_N \quad &= \quad e^{-i\eta mN} S_N\left(1, \frac{2\mathrm{i}\eta - 2N + N_f}{2}, \frac{N_f - 2N - 2\mathrm{i}\eta}{2}\right) \\ &= \quad e^{-i\eta mN} \prod_{j=0}^{N-1} \frac{\Gamma(\mathrm{i}\eta+1+N_f/2-N+j)\Gamma(-\mathrm{i}\eta+1+N_f/2-N+j)\Gamma(j+2)}{\Gamma(-N+N_f+1+j)}. \end{aligned} \tag{3.3}$$

Once again, the partition function can be written in terms of Barnes G-functions

$$Z_N = e^{-i\eta mN} \frac{G(\mathrm{i}\eta+1+N_f/2)G(-\mathrm{i}\eta+1+N_f/2)G(-N+N_f+1)G(N+2)}{G(\mathrm{i}\eta+1+N_f/2-N)G(-\mathrm{i}\eta+1+N_f/2-N)G(N_f+1)G(2)}, \tag{3.4}$$

which, as expected, coincides with (2.7). The expression simplifies when $\eta = 0$, giving

$$Z_{N_f}^{U(N)}(\eta=0) = \frac{G^2(1+N_f/2)G(-N+N_f+1)G(N+2)}{G^2(1+N_f/2-N)G(N_f+1)}. \tag{3.5}$$

In the $\eta = 0$ case, as seen also from the condition (3.2), the result holds for $N_f \geq 2N-1$, a well-known result [12]. In the large N and N_f limit, therefore, this implies a lower bound for the Veneziano parameter $\zeta = N_f/N > 2$. Now, one can immediately study the large $N_f \to \infty$ and $N \to \infty$ behavior in (3.5), directly using the original expansion by Barnes

(3.6)

$$\log G(z+1) = \frac{1}{2} - \log A + \frac{z}{2}\log 2\pi + \left(\frac{z^2}{2} - \frac{1}{12}\right)\log z - \frac{3z^2}{4} + \sum_{k=1}^{N} \frac{\mathrm{B}_{2k+2}}{4k(k+1)z^{2k}} + O\left(\frac{1}{z^{2N+2}}\right),$$

where the constant A is the Glaisher–Kinkelin constant. The solution can be given in terms of N and the Veneziano parameter, in which case the partition function reads

$$Z(N,\zeta,\eta) = \frac{G(1+i\eta+\frac{\zeta N}{2})G(1-i\eta+\frac{\zeta N}{2})G(((\zeta-1)N+1)G(N+2)}{G(i\eta+(\zeta/2-1)N+1)G(-i\eta+(\zeta/2-1)N+1)G(\zeta N+1)}. \tag{3.7}$$

For $\eta = 0$ we can also for example study the first convergent case, given by $N_f = 2N+1$ or, more generally, $N_f = 2N+k$ with $k \geq 1$ and finite, which gives

$$Z_{2N+1}^{U(N)}(\eta=0) = \frac{G^2(1+N+\frac{k}{2})G(N+k+1)G(N+2)}{G^2(1+k/2)G(2N+1+k)},$$

this covers an arbitrary large number of cases (indexed by k), and the Veneziano parameter is always 2. Now, for $\eta = 0$ and taking into account the asymptotics of the Barnes G-function, it follows that, for the double scaling limit $N \to \infty$ and $N_f \to \infty$ with $\zeta = N_f/N = \text{cte}$, if we write the free energy as $F_N \equiv \ln Z_N$, what would be the leading term in the free energy

$$\frac{N^2}{2}\log N, \tag{3.8}$$

always cancels out and it is not present. This holds because, if we denote by z_i the terms $1+z_i$ in the arguments of the different G-Barnes in (3.7) with $\eta = 0$[6], then it holds that

$$\sum_{i=1}^{4} z_i^2 - \sum_{i=5}^{7} z_i^2 = \sum_{i=1}^{4} z_i - \sum_{i=5}^{7} z_i = 2N+1. \tag{3.9}$$

The final term that comes from the $-3z^2/4$ piece in (3.6) cancels completely due to (3.9) and the same happens, partially, for the term that results from the $(z^2/2)\log z$ piece in (3.6). The (3.8) behavior that would follow from this piece cancels, and it only remains a N^2 leading term that is multiplied by a Veneziano parameter dependent factor. All together we have, for $N \to \infty$

$$F_N \sim N^2 f(\zeta) + N\log N + N\left(\log 2\pi - \frac{3}{2}\right) + \frac{5}{12}\log N,$$

where

$$f(\zeta) = \frac{\zeta^2}{2}\log\frac{\zeta}{2} + (\zeta-1)^2\log(\zeta-1) - 2\left(\frac{\zeta}{2}-1\right)^2\log\left(\frac{\zeta}{2}-1\right) - \zeta^2\log\zeta. \tag{3.10}$$

Note that for the particular case of $\zeta = 2$ all the terms in (3.10) cancel, with the exception of the last one and then $f(\zeta) = -4\log 2$. In general, while naively at first it may seem that $f(\zeta)$ vanishes for large ζ, it turns out that it is well approximated by $f(\zeta) \simeq -2\zeta\log 2$ for large ζ.

Recall that the free energy of the larger family of 3d $T_\rho^{\rho'}[SU(N)]$ theories, at leading order, has been studied in [14, 15] and, more recently, from the point of view of gravitational duals, in [16] where it is shown that the leading term is of the form (3.8). This property also holds for the simpler $T[SU(N)]$ theory, namely the $SU(N)$ theory with N_f flavours of different masses m_l with $l = 1, ..., N_f$ [14, 15], and we have just studied the case when all masses are equal, which leads to the cancellation of such a leading term. For a very recent discussion of other 3d SYM theories with similar behavior to the one obtained here, see [36].

[6]From $i = 1$ to 4 it denotes the arguments in the numerator and from 5 to 7 the ones in the denominator.

3.2. $O(2N), O(2N+1)$ **and** $Sp(2N)$ **cases.** Now, setting the mass and the FI parameter $\eta = m = 0$, the explicit expression of the free energy when the gauge group are the orthogonal and symplectic groups can be obtained, again with the same Selberg integral, just by considering another change of variables. Indeed, with the change of variables [2, Ex. 4.1.3]

$$t_l = \frac{2}{\cosh(s_l/2)+1}$$

we have that

$$\begin{aligned} S_N(\lambda_1,\lambda_2,\lambda) &= 2^{-N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{i=1}^{N}\frac{\left(\sinh^2(s_l/2)\right)^{\lambda_2+1/2}ds_l}{\left(\cosh^2(s_i/2)\right)^{\lambda_1+\lambda_2+2+2\lambda(N-1)}} \\ &\times\prod_{i<j}\left(\sinh(\frac{1}{2}(s_i-s_j))\sinh(\frac{1}{2}(s_i+s_j))\right)^{2\lambda}. \end{aligned} \tag{3.11}$$

Recall that the Vandermonde determinant of the matrix model, in case of the orthogonal and symplectic gauge groups is the corresponding hyperbolic version of the Haar measure, explicitly given by:

$$\Delta^2_{O(2N)}(e^{is}) = \prod_{i<j}\left(2\sinh(\frac{1}{2}(s_i-s_j))2\sinh(\frac{1}{2}(s_i+s_j))\right)^2,$$

$$\begin{aligned} \Delta^2_{O(2N+1)}(e^{is}) &= \prod_{i<j}\left(2\sinh(\frac{1}{2}(s_i-s_j))2\sinh(\frac{1}{2}(s_i+s_j))\right)^2\prod_{i=1}^{N}\sinh^2(s_l/2), \\ \Delta^2_{Sp(2N)}(e^{is}) &= \prod_{i<j}\left(2\sinh(\frac{1}{2}(s_i-s_j))2\sinh(\frac{1}{2}(s_i+s_j))\right)^2\prod_{i=1}^{N}\sinh^2(s_l). \end{aligned}$$

Thus, we find that, in terms of the S_N given by (3.1), the partition functions read:

$$\begin{aligned} Z^{O(2N)}_{N_f} &= 2^{2N(N-(N_f-3)/2)}S_N((N_f+1)/2-2N,-1/2,1) \\ Z^{O(2N+1)}_{N_f} &= 2^{2N(N-(N_f-3)/2)}S_N((N_f-1)/2-2N,+1/2,1) \\ Z^{Sp(2N)}_{N_f} &= S_N((N_f-3)/2-2N,+1/2,1), \end{aligned}$$

where for the symplectic case we have used a half-angle formula for the sinh function in (3.11). The partition functions in terms of the G Barnes function are then given by

$$\begin{aligned} Z^{O(2N)}_{N_f} &= 2^{2N(N-(N_f-3)/2)}\frac{G\left((N_f+1)/2-N+1\right)G(N+1/2)G(N+2)G\left(N_f/2-N\right)}{G\left((N_f+1)/2-2N+1\right)G(1/2)G\left(N_f/2\right)}, \\ Z^{O(2N+1)}_{N_f} &= 2^{2N(N-(N_f-3)/2)}\frac{G\left((N_f-1)/2-N+1\right)G(N+3/2)G(N+2)G\left(N_f/2-N+1\right)}{G\left(N_f/2+1\right)G(3/2)G\left(N_f/2+1/2\right)}, \\ Z^{Sp(2N)}_{N_f} &= \frac{G\left((N_f-3)/2-N+1\right)G(N+3/2)G\left(N_f/2-N\right)}{G\left(N_f/2\right)G\left((N_f-3)/2-2N+1\right)G(3/2)}. \end{aligned}$$

3.3. **Mellin-Barnes representation for** $SU(N)$**.** To conclude this Section, notice that the $SU(N)$ case could also be computed by integrating the $U(N)$ result over the FI parameter (as this is equivalent to introduce a Dirac delta in the matrix model). We start by rewriting the expression (2.7) in terms of Gamma functions for the part containing the FI parameter. That is

$$Z_N = e^{-imN\eta}\prod_{j=1}^{N}\left|\Gamma\left(\frac{N_f}{2}-j+i\eta+1\right)\right|^2\frac{G\left(N+2\right)G(N_f-N+1)}{G(N_f+1)}, \tag{3.12}$$

which follows immediately by repeatedly using $G(z+1) = \Gamma(z)G(z)$. First, we analyze the $SU(2)$ case. In the massless case, we can directly use first Barnes lemma

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(a+x)\Gamma(b+x)\Gamma(c-x)\Gamma(d-x)\,dx = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)},$$

which is an extension of the beta integral and equivalent to Gauss summation of the hypergeometric function. Then, normalizing by 2π to account for the Dirac delta representation as a Fourier transform of the identity, we have

$$\begin{aligned} Z_{SU(2)} &= \frac{G(4)\,G(N_f-1)}{(2\pi)\,G(N_f+1)}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{N_f}{2}+i\eta\right)\right|^2\left|\Gamma\left(\frac{N_f}{2}+i\eta-1\right)\right|^2 d\eta \\ &= \frac{G(4)\,G(N_f-1)\Gamma(N_f)\Gamma(N_f-1)^2\Gamma(N_f-2)}{G(N_f+1)\Gamma(2N_f-2)}. \end{aligned}$$

Notice that the expression diverges for $N_f = 1$ and $N_f = 2$ and indeed these two cases correspond to *bad* theories. Of course, for this case the direct computation of the matrix model (1.1) with the $\delta(x_1+x_2)$ insertion is direct as well, with the resulting one dimensional integral, giving:

$$\widetilde{Z}_{SU(2)} = \int_{-\infty}^{\infty}\frac{4\sinh^2 x}{(2\cosh(x/2))^{2N_f}}dx, \tag{3.13}$$

and, using for example (1.3), one checks that indeed $\widetilde{Z}_{SU(2)} = Z_{SU(2)}$. The massive version of (3.13) is also solvable (for example, using the results in [13]) and can be related to Wilson loops in a $U(1)$ theory with N_f hypermultiplets of mass m and N_f hypermultiplets of mass $-m$, but this will be discussed elsewhere.

From the expression (3.12), it follows that the partition function for the $SU(N)$ theory admits a one-dimensional Mellin-Barnes type of integral representation

$$Z_{SU(N)} = \frac{G(N+2)\,G(N_f-N+1)}{G(N_f+1)}\int_{-\infty}^{\infty}e^{-imN\eta}\prod_{j=1}^{N}\Gamma\left(\frac{N_f}{2}-j+i\eta+1\right)\Gamma\left(\frac{N_f}{2}-j-i\eta+1\right)d\eta, \tag{3.14}$$

with a Fourier kernel also, in the massive case. Both asymptotics and a full analytical solution are possible but not entirely immediate (a formula for generic values of all parameters ends up being quite involved), requiring an analysis in itself, and hence it will be presented elsewhere.

4. On exponential asymptotics and duality

The presence of the FI parameter has an interesting implication: the expressions for the free energies and the Wilson loops, given above, are in terms of G-Barnes or Gamma functions of *complex* arguments. This leads naturally to consider the behavior of these functions in the whole complex plane, where a richer behavior, involving Stokes lines [35, 19], is well-known to emerge.

Let us remind first that $G(z+1)$ and $\Gamma(s)$ (we will directly look at the asymptotic expansion of their logarithm) has Stokes lines at $z = \pm\pi/2$. Thus, looking at the logarithm of the solution (2.7), it appears that for certain values of the rank of the gauge group N and the number of flavours N_f (and, in the case of the Wilson loops, the size of the representation), we will have the appearance of exponentially small contributions in the asymptotic expansions of the observables. Notice that these are not phase transitions or crossovers within eventual different regimes of the theory, since the controlling parameters are number of flavours and colors.

We focus on the $U(N)$ free energy, given by the logarithm of (3.4). Its analysis immediately follows from considering the mathematical results on the G Barnes function [21, 19]. The exponentially improved asymptotic expansion of the G-Barnes function reads [19]

$$\begin{aligned}\log G(z+1) \sim \frac{1}{4}z^2 + z\log\Gamma(z+1) - \left(\frac{1}{2}z(z+1)+\frac{1}{12}\right)\log z - \log A \\ + \sum_{k=1}^{\infty} S_k(\theta)\, e^{\pm 2\pi i k z} + \sum_{n=1}^{\infty} \frac{B_{2n+2}}{2n(2n+1)(2n+2)z^{2n}},\end{aligned} \tag{4.1}$$

where

$$S_k(\theta) = \begin{cases} 0 & \text{if } |\theta| < \frac{\pi}{2} \\ \mp \frac{1}{2}\frac{1}{2\pi i k^2} & \text{if } \theta = \pm\frac{\pi}{2} \\ \mp \frac{1}{2\pi i k^2} & \text{if } \frac{\pi}{2} < |\theta| < \pi, \end{cases} \tag{4.2}$$

and $\theta = \arg z$. The upper or lower sign is taken according to z being in the upper or lower half-plane. The term with $S_k(\theta)$ describes the Stokes singularities and the rest of (4.1) is the asymptotic expansion of the G Barnes function, as given in [37]. Note that this is not the original asymptotic expansion by Barnes, used above, but, using the asymptotics for the $\log\Gamma(z+1)$ in (4.1) (including exponentially small contributions) one can write down the analogous result for the Barnes form above (3.6). At any rate, as will be seen below, we crucially need both asymptotics for our analysis of the free energy.

The four $G(1+z)$ functions of the free energy (taking the logarithm of (2.7)) with complex argument, have a z variable given by:

$$\begin{aligned} z_{1,\pm} &= \pm \mathrm{i}\eta + N_f/2 - N, \\ z_{2,\pm} &= \pm \mathrm{i}\eta + N_f/2. \end{aligned}$$

The Stokes lines are located at $\pm\pi/2$, we need to look when the real part of the arguments becomes 0 and/or negative. A null or negative real part argument can only physically happen for the $z_{1,\pm}$ case above. The first case, which is right at the Stokes line, corresponds to $N_f = 2N$ which is a self-dual case in terms of dualities (see below). The first actual crossing of a Stokes line occurs for $N_f = 2N - 1$, the ugly case, and the rest is then for

$$N_f < 2N - 1,$$

which is well-known to correspond to the so-called *bad* theories. Therefore, we discuss below this eventual crossing of the Stokes line also in the context of the analysis of *bad*, *good* and *ugly* theories [11, 12] and dualities below.

However, we need the asymptotics of the Gamma function too, whose Stokes phenomena is similar to that of the G-Barnes function above, with the Stokes lines also at $\pm\pi/2$. One difference is the different decay of the $S_k(\theta)$ coefficients (Stokes multipliers), where the quadratic decay in the G-Barnes is a linear decay in the Gamma function case [20]. More crucially, there is a sign difference in the respective Stokes multipliers, as we show in what follows. For the Gamma function the following asymptotic expansion holds as $z \to \infty$

$$\log\Gamma^*(z) \sim \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} - \begin{cases} 0 & \text{if } |\theta| < \frac{\pi}{2} \\ \frac{1}{2}\log(1-e^{\pm 2\pi i z}) & \text{if } \theta = \pm\frac{\pi}{2} \\ \log(1-e^{\pm 2\pi i z}) & \text{if } \frac{\pi}{2} < |\theta| < \pi, \end{cases} .$$

The expansion of the logarithm brings the asymptotics in the same form as above

$$\log\Gamma^*(z) \sim \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} + \sum_{k=1}^{\infty} \widetilde{S}_k(\theta)\, e^{\pm 2\pi i k z}, \tag{4.3}$$

in the sector $|\arg z| \leq \pi - \delta < \pi$ for any $0 < \delta \leq \pi$ with[7]

$$\widetilde{S}_k\left(\theta\right) = \begin{cases} 0 & \text{if } |\theta| < \frac{\pi}{2} \\ \frac{1}{2k} & \text{if } \theta = \pm\frac{\pi}{2} \\ \frac{1}{k} & \text{if } \frac{\pi}{2} < |\theta| < \pi, \end{cases} \quad , \tag{4.4}$$

where the usual definition

$$\Gamma^*(z) = \frac{\Gamma(z)}{\sqrt{\pi} z^{z-1/2} e^{-z}},$$

was used. In addition to the different decay, the sign difference between $\widetilde{S}_k\left(\theta\right)$ and $S_k\left(\theta\right)$ is crucial with regards to the cancellation of exponentially small terms in the asymptotics of the observables. Notice also the difference in how the variable is written in the argument of the functions in (4.3). Thus, the relevant variable indicating a possible Stokes line crossing, in this case, is then

$$\widetilde{z}_{1,\pm} = \pm \mathrm{i}\eta + N_f/2 - N + 1.$$

4.1. **Absence of exponentially small contributions for good and ugly theories.** Taking into account the specific form of the Stokes multipliers parts in (4.1) and (4.3), we show now that there will be no exponentially small contributions for the case of *ugly* and *bad* theories. Since we want to use the asymptotic result for $z \to \infty$ we then need to take the large η limit.

For example in the *good* theory case which is self-dual, $N_f = 2N$ then $z_{1,\pm} = \pm \mathrm{i}\eta$ and we are on the two Stokes lines always for the two G-Barnes functions in the denominator of (2.7). We focus only on the exponentially small contributions to the free energy. That is, in the piece with the Stokes multipliers in (4.1) for the two G-Barnes functions $\log G(i\eta + 1) + \log G(-i\eta + 1)$. We have that the two set of contributions cancel each other

$$F_{N,\text{good}}^{\text{Stokes}} \sim -\frac{1}{2} \sum_{k=1}^{\infty} \frac{e^{-2\pi k\eta} - e^{-2\pi k\eta}}{2\pi i k^2} = 0.$$

Notice that this result and the one below also holds for finite N, however for the asymptotics of the rest of G Barnes functions in (2.7) one needs to take the large N limit. With regards to the Gamma asymptotics, we have not reach the Stokes line for these values of the parameter, because $\widetilde{z}_{1,\pm} = \pm \mathrm{i}\eta + 1$ in this case.

In the *ugly* case $N_f = 2N - 1$ then $z_{1,\pm} = \pm \mathrm{i}\eta - 1/2$ and we have crossed the Stokes line in the G-Barnes function asymptotics. As above, we have cancellation, due to the symmetric way in which the G-Barnes function appear in the partition function. Focussing on the piece with the Stokes multipliers in (4.1), for the two G-Barnes functions $\log G(i\eta + 1/2) + \log G(-i\eta + 1/2)$ we have

$$F_{N,\text{ugly}}^{\text{Stokes}} \sim -\sum_{k=1}^{\infty} \frac{e^{-2\pi k\eta + \pi i k} - e^{-2\pi k\eta - \pi i k}}{2\pi i k^2} = 0.$$

In addition, this is the last case where the Gamma function asymptotics will not contribute, since the Stokes line is not yet reached, as $\widetilde{z}_{1,\pm} = \pm \mathrm{i}\eta + 1/2$.

[7]Note that, with regards to the location of Stokes lines, that the asymptotics of the Gamma function is with variable z whereas of the G-Barnes function is $z + 1$.

4.2. **Exponentially small contributions in bad theories.** For bad theories, we have $N_f = 2N - 2 - l$ with $l = 0, 1, 2, ...$ then $z_{1,\pm} = \pm i\eta - 1 - l/2$ for the G-Barnes function and $\widetilde{z}_{1,\pm} = \pm i\eta - l/2$ for the corresponding Gamma function asymptotics, then

$$F_{N,\text{bad}}^{\text{Stokes}} = -\sum_{k=1}^{\infty} \frac{e^{-2\pi k\eta}\sin(\pi kl)}{2\pi ik^2} - \sum_{k=1}^{\infty} \frac{2e^{-2\pi k\eta}\cos\left[2\pi k\left(1+l/2\right)\right]}{k} = -2\left(-1\right)^l \sum_{k=1}^{\infty} \frac{e^{-2\pi k\eta}}{k},$$

and therefore the sign of the contribution is opposite according to the parity of N_f. Therefore, we have seen that only the exponentially small terms in the asymptotics of the Gamma function eventually contribute. Another way of obtaining this result, would have been to directly invoke the equivalent expression (3.12) for the partition function, but we it is illustrative to obtain it from the asymptotics of the G-Barnes function.

Now, localization on S^3 [31] permits also to consider the case $\eta \in \mathbb{C}$. If we write $\eta = \eta_R + i\eta_I$ then, there are clearly many more possibilities of crossing Stokes lines and the four G-Barnes functions can now give corresponding exponentially small contributions, because the crossings are now determined by:

$$\begin{aligned} \pm\eta_I + N_f/2 - N &\leq 0 \\ \pm\eta_I + N_f/2 &\leq 0, \end{aligned}$$

for the Stokes multipliers in (4.1) and

$$\begin{aligned} \pm\eta_I + N_f/2 - N &\leq 0 \\ \pm\eta_I + N_f/2 &\leq 0, \end{aligned}$$

for the ones in (4.3), corresponding to the Gamma function.

Besides, we saw above that this case is equal to an unnormalized (not divided by the partition function) Wilson loop with a rectangular representation of the type (N^{η_I}). Indeed the case of Wilson loops leads to a different situation. This will be discussed in detail elsewhere, but note, for example focussing on the result for the symmetric representations (2.11), that while the term $\Gamma\left(N_f/2 - N - i\eta\right)$ will work as above and only have exponentially small terms for *bad* theories. The term in the numerator, due to its part with r, will lead to such contributions even for *good* theories (in particular, for $r = 1$ it will affect the *ugly* theory and for $r > 1$, *good* theories).

4.3. **Duality.** Some simple tests of Seiberg duality can be quickly carried out. This is an additional test on the formula (2.7), but it is also relevant from the point of view of the asymptotics just presented. We consider the generic case given by $N_f = 2N + k$ for a general integer value of k. We start first with $k = -1$. Then, theory $U(N)$ with $N_f = 2N - 1$ is in the *ugly* class, containing a decoupled free sector, generated by BPS monopole operators of dimension $\frac{1}{2}$. It is known that the rest is dual to the IR-limit of the $U(N-1)$ gauge theory with $N_f = 2N - 1$, and thus a *good* theory. Using (2.7) and again $G(z+1) = \Gamma(z)G(z)$ one quickly finds that

$$Z_N(N_f = 2N-1) = e^{-im\eta} N\Gamma(-i\eta + 1/2)\Gamma(i\eta + 1/2) Z_{N-1}(N_f = 2N-1), \tag{4.5}$$

the N arises due to the fact that we dropped the $N!$ normalizing factor in (1.1) and denoted the resulting partition function by Z_N, restoring it, we obtain that for (1.1) it holds that

$$Z_{N_f=2N-1}^{U(N)} = Z_{N_f=2N-1}^{U(N-1)} \frac{\pi e^{-im\eta}}{\cosh(\pi\eta)}, \tag{4.6}$$

and we have the expected duality between the $U(N)$ and $U(N-1)$ theories, together with the expected appearance of a free hypermultiplet. This is the case where the duality is between a *good* and *ugly* theory, whereas the rest will be between *good* and *bad* theory. There were no exponentially small contributions for both theories and indeed the free hypermultiplet part, given by the Gamma factors in (4.5) do not add any such contribution. This is different from what occurs in the rest of the duality cases.

If we take $k = 1$ instead of $k = -1$ then we have to obtain the same duality, but starting from the good theory side. An immediate computation shows that this is indeed the case, giving again (4.6). The self-dual case $N_f = 2N$ is evident and the rest is, naively, between *good* and *bad* theories, which corresponds to starting with a $U(N)$ theory with $N_f > 2N+1$. For example, for $U(N)$ theory with $N_f = 2N+2$, we have

$$Z_N(N_f = 2N+2) = \frac{e^{2im\eta}}{(N+2)(N+1)} \left|\Gamma(-i\eta+1)\right|^{-2} \left|\Gamma(-i\eta)\right|^{-2} Z_{N+2}(N_f = 2N+2), \tag{4.7}$$

then

$$Z^{U(N+2)}_{N_f=2N+2} = Z^{U(N)}_{N_f=2N+2} \frac{\pi^2 e^{-2im\eta}}{\sinh^2(\pi\eta)}.$$

Thus, this is the duality between a good theory on the l.h.s. of (4.7) and a bad theory on the r.h.s. The latter we have seen that has exponentially small contributions in its asymptotics whereas the former does not. This difference is accounted for by a part of the factorized free sector and, indeed, note the $|\Gamma(-i\eta)|^{-2}$ term in (4.7). This is exact since the bad theory on the r.h.s is precisely the one analyzed above for $l = 0$ whose exponentially small asymptotics is precisely given by $\Gamma(i\eta)$ and $\Gamma(-i\eta)$, since $\widetilde{z}_{1,\pm} = \pm i\eta$ in that case.

Thus, taking the log and passing the factors corresponding to $|\Gamma(-i\eta)|^{-2}$ to the l.h.s., we see that the two exponentially small contributions on both sides of the equality are identical (alternatively, the factor cancels the exponentially small contributions in the bad theory, leaving no contributions). Thus, the asymptotics behavior explained above is completely consistent with the explicit duality check here.

The duality check above can be extended to the generic cases of even and odd number of flavours $N_f = 2N+2k$ for $k = 1, 2, 3, ...$ and $N_f = 2N+2k+1$ for $k = 0, 1, 2, 3, ...$ (the negative k gives the same duality as its positive counterpart). We obtain:

$$\begin{aligned} Z^{U(N+2k)}_{N_f=2N+2k} &= Z^{U(N)}_{N_f=2N+2k} e^{-2ikm\eta} \prod_{j=1-k}^{k} \left|\Gamma\left(j+i\eta\right)\right|^2 = Z^{U(N)}_{N_f=2N+2k} \left(\frac{\pi e^{-im\eta}}{\sinh(\pi\eta)}\right)^{2k}, \\ Z^{U(N+2k+1)}_{N_f=2N+2k+1} &= Z^{U(N)}_{N_f=2N+2k+1} e^{-i(2k+1)m\eta} \prod_{j=-k}^{k} \left|\Gamma\left(j+\frac{1}{2}+i\eta\right)\right|^2 = Z^{U(N)}_{N_f=2N+2k+1} \left(\frac{\pi e^{-im\eta}}{\cosh(\pi\eta)}\right)^{2k+1}. \end{aligned}$$

5. Outlook

We expect to further study the asymptotics together with the duality, including further discussion on the case of Wilson loops and the setting where a gauge-R Chern-Simons term is present, characterized by an additional imaginary part in the FI parameter. The asymptotics in this case will admit even more possibilities and it should be possible to also look at it from the point of view of Borel transforms. The Mellin-Barnes type of integral given for the $SU(N)$ theory can also be exploited for both a full analytical solution and for a study of asymptotics as well. We expect to present all this in a future work.

We conclude by briefly commenting on the fact that the matrix model studied here not only is related to an extended Selberg integral [27, 30], but also it has several equivalent representations [27, 30], a well-known result in the context of Berezin quantization, whose original connections with matrix models precisely relied heavily on matrix integration identities [38]. In this way, the matrix model for the partition function without a FI term also appears in the (Berezin) quantization analysis characterizing the Hilbert space of a collective field theory of the singlet sector of the symplectic $Sp(2N)$ sigma model, of importance in the study of dS/CFT correspondence

[39, 40]. In that setting, it computes the size of the Hilbert space. In principle the Wilson loop studied here will also have an interpretation in this Hilbert space picture.

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