TOEPLITZ MINORS FOR SZEGŐ AND FISHER-HARTWIG SYMBOLS

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Abstract. We study minors of Toeplitz matrices of both finite and large dimension, comprising also the case of symbols with Fisher-Hartwig singularities. We express the minors in terms of specializations of symmetric polynomials. Several implications of this formulation are presented, including explicit formulas for a Selberg-Morris integral with two Schur polynomials and for the specialization of skew-Schur polynomials. For the latter result, the inverse of a Toeplitz matrix with a pure Fisher-Hartwig singularity is computed, using both our results on minors and the Duduchava-Roch formula.

1. Introduction

The study of Toeplitz matrices has attracted interest for many decades and is a subject of remarkable mathematical relevance, also enjoying a large number of applications in a wide-ranging number of fields [22, 6, 7, 21]. For the more specific topic of Toeplitz determinants, [15] is a detailed review. We begin by introducing a few definitions and quoting some fundamental and classical results.

Let $f(e^{i\theta}) = \sum_{k\in\mathbb{Z}} d_k e^{ik\theta}$ be an integrable function on the unit circle $\mathbb{T}$. The Toeplitz matrix of order N generated by f is the matrix

$$T_N(f) = (d_{j,k})_{j,k=1}^N = (d_{j-k})_{j,k=1}^N.$$

We refer to f as the symbol of the matrix, and we omit the subindex N to denote the infinite matrix $T(f) = (d_{j-k})_{j,k\geq 1}$. We denote

$$D_N(f) = \det T_N(f).$$

A main result of the theory of Toeplitz matrices is the strong Szegő limit theorem, that describes the behaviour of these determinants as N grows to infinity.

Theorem (Szegő). *Let $f(e^{i\theta}) = \exp(\sum_{k\in\mathbb{Z}} c_k e^{ik\theta})$, with $\sum_{k\in\mathbb{Z}} |c_k| < \infty$ and $\sum_{k\in\mathbb{Z}} |k||c_k|^2 < \infty$. Then*

$$\lim_{N\to\infty} \left(D_N(f)/e^{Nc_0}\right) = \exp\left(\sum_{k=1}^{\infty} k c_k c_{-k}\right).$$

An important aspect of Toeplitz matrices is their intimate relationship with random matrix theory. This connection is established by means of the Heine (or Heine-Szegő) identity, which gives an expression of the determinant $D_N(f)$ as an integral over the unitary group. That is, as a unitary matrix model. It reads

$$D_N(f) = \frac{1}{N!}\frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{j=1}^N f(e^{i\theta_j})\prod_{1\leq j<k\leq N}|e^{i\theta_j}-e^{i\theta_k}|^2 d\theta_1\ldots d\theta_N = \int_{U(N)} f(M)dM,$$

where dM denotes the normalized Haar measure on $U(N)$, and the second identity follows from the Weyl integration formula.

Around 15 years ago, Bump and Diaconis [11] obtained generalizations of these results for minors of Toeplitz matrices (*Toeplitz minors* in the following). These correspond to matrices of the form $(d_{p_j-q_k})_{j,k}$, where p_j and q_k are sequences of integers that verify $p_k = q_k = k$ for large enough values of k. These sequences can be assumed to be strictly increasing without loss of generality, except for a possible change of sign of the determinant. Hence we can write such matrices as

$$T_N^{\lambda,\mu}(f) = (d_{j-\lambda_j-(k-\mu_k)})_{j,k=1}^N,$$

where λ and μ are integer partitions. That is, finite sequences of non-increasing positive integers (we use in the following standard facts and notation from the theory of symmetric functions, which can be found in [26, 31], for instance). We denote

$$D_N^{\lambda,\mu}(f) = \det T_N^{\lambda,\mu}(f).$$

As described in [11], these determinants are in fact minors of the corresponding Toeplitz matrices, obtained according to the following procedure:

- Strike the first $|\lambda_1 - \mu_1|$ columns or rows of $T_{N+\max\{\lambda_1,\mu_1\}}(f)$, depending on whether $\lambda_1 - \mu_1$ is greater or smaller than zero, respectively.
- Keep the first row of the matrix, and strike the next $\lambda_1 - \lambda_2$ rows. Keep the next row, and strike the next $\lambda_2 - \lambda_3$ rows. Continue until striking $\lambda_{l(\lambda)} - \lambda_{l(\lambda)+1} = \lambda_{l(\lambda)}$ rows ($l(\lambda)$ denotes the length of the partition λ, as usual).
- Repeat the previous step on the columns of the matrix with μ in place of λ. Note that we have striked exactly $\max\{\lambda_1, \mu_1\}$ rows and columns from the original matrix.

The generalizations of Heine identity and the strong Szegő limit theorem for Toeplitz minors read as follows.

Theorem 1 (Bump, Diaconis)**.** *Let f be an integrable function on $\mathbb{T}$, and let λ, μ be partitions with $l(\lambda), l(\mu) \leq N$. Then*

$$D_N^{\lambda,\mu}(f) = \int_{U(N)} \overline{s_\lambda(M)} s_\mu(M) f(M) dM =$$

$$\frac{1}{N!}\frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} s_\lambda(e^{-i\theta_1},\ldots,e^{-i\theta_N}) s_\mu(e^{i\theta_1},\ldots,e^{i\theta_N}) \prod_{j=1}^N f(e^{i\theta_j}) \prod_{1\leq j<k\leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \ldots d\theta_N,$$

where s_λ, s_μ are Schur polynomials [26], defined on $U(N)$ as[1] $s_\lambda(M) = s_\lambda(e^{i\theta_1}, \ldots, e^{i\theta_N})$*, where the $e^{i\theta_j}$ are the eigenvalues of M.*

Theorem 2 (Bump, Diaconis)**.** *Let $f(e^{i\theta}) = \exp(\sum_{k\in\mathbb{Z}} c_k e^{ik\theta})$, with $\sum_{k\in\mathbb{Z}} |c_k| < \infty$ and $\sum_{k\in\mathbb{Z}} |k||c_k|^2 < \infty$, and suppose λ and μ are partitions of weights n and m respectively. Then*

$$\lim_{N\to\infty}\left[\frac{D_N^{\lambda,\mu}(f)}{D_N(f)}\right] = \sum_{\phi\vdash n}\sum_{\psi\vdash m} \chi_\phi^\lambda\, \chi_\psi^\mu\, z_\phi^{-1} z_\psi^{-1} \Delta(f,\phi,\psi), \tag{1}$$

where the sum runs over all the partitions ϕ of n and ψ of m, the terms z_ϕ, z_ψ are the orders of the centralizers of the equivalence classes of the symmetric groups S_n, S_m indexed by ϕ and ψ respectively, the functions χ^λ, χ^μ are the characters associated to the irreducible representations of S_n and S_m indexed by λ and μ respectively, and

$$\Delta(f,\phi,\psi) = \prod_{k=1}^\infty \begin{cases} k^{n_k} c_{-k}^{n_k-m_k} m_k! L_{m_k}^{(n_k-m_k)}(-kc_k c_{-k}), & \text{if } n_k \geq m_k \\ k^{m_k} c_k^{m_k-n_k} n_k! L_{n_k}^{(m_k-n_k)}(-kc_k c_{-k}), & \text{if } n_k \leq m_k \end{cases}.$$

Above, the coefficients n_k, m_k correspond to the partitions $\phi = (1^{n_1}2^{n_2}\ldots)$ and $\psi = (1^{m_1}2^{m_2}\ldots)$ in their frequency notation, and $L_n^{(a)}$ are the Laguerre polynomials [32].

[1] We abuse notation here. We assume it is clear, in each case, when the expression $f(M)$ should be read as $\prod_j f(e^{i\theta_j})$ (i.e. when f is a function on $\mathbb{T}$), and when it should be read as $f(e^{i\theta_1}, \ldots, e^{i\theta_N})$ (i.e. when f is a symmetric function).

λ	μ	$\lim_{N\to\infty} D_N^{\lambda,\mu}(f)/D_N(f)$	λ	μ	$\lim_{N\to\infty} D_N^{\lambda,\mu}(f)/D_N(f)$
$\varnothing$	(1)	c_1	$\varnothing$	(2)	$\frac{1}{2}c_1^2 + c_2$
$\varnothing$	$(1,1)$	$\frac{1}{2}c_1^2 - c_2$	$\varnothing$	(3)	$\frac{1}{6}c_1^3 + c_1c_2 + c_3$
$\varnothing$	$(1,1,1)$	$\frac{1}{6}c_1^3 - c_1c_2 + c_3$	$\varnothing$	$(2,2)$	$\frac{1}{12}c_1^4 - c_1c_3 + c_2^2$

λ	μ	$\lim_{N\to\infty} D_N^{\lambda,\mu}(f)/D_N(f)$
$(1,1)$	$(1,1)$	$\frac{1}{4}c_{-1}^2c_1^2 + c_{-1}c_1 - \frac{1}{2}c_{-2}c_1^2 - \frac{1}{2}c_{-1}^2c_2 + c_{-2}c_2 + 1$
(1)	(3)	$\frac{1}{6}c_{-1}c_1^3 + \frac{1}{2}c_1^2 + c_{-1}c_1c_2 + c_2 + c_{-1}c_3$

Table 1. Some values of the formula (1).

If the partition λ is empty, the formula (1) simplifies to

$$\lim_{N\to\infty}\left[\frac{D_N^{\varnothing,\mu}(f)}{D_N(f)}\right] = \sum_{\psi\vdash m}\chi_\psi^\mu \prod_{k\geq 1}\frac{c_k^{m_k}}{m_k!}.$$

Note also that both the product above and the one appearing in the factor $\Delta(f,\phi,\psi)$ in theorem 2 are actually finite, since only a finite number of n_k, m_k are distinct from zero for each ϕ,ψ. The formula (1) can be implemented in MatLab, leading to quick evaluations for values of, say, $|\lambda|, |\mu| = 15$. Table 1 shows some of these values for particular choices of λ and μ.

In this paper we study the minor $D_N^{\lambda,\mu}(f)$ at finite N as well, giving expressions in terms of symmetric functions. We extend the previous results by considering more general symbols, including singularities, also for both finite and large N. As an application of this result, we give an evaluation of a Selberg integral of the Morris type [18], generalized with two Schur polynomials in the integrand, in terms of a specialized skew-Schur polynomial. We also study the inverse of a Toeplitz matrix with a Fisher-Hartwig (FH) singularity [19], leading to an evaluation of a specialization of certain skew-Schur polynomials.

In addition to the works following [11], such as [13, 14, 25, 34], other aspects of minors of Toeplitz matrices have been discussed in [24, 28], also in relationship with specializations of Schur and skew-Schur functions.

The paper is organized as follows. In the next Section we present our first result, which gives an expression for a generic Toeplitz minor $D_N^{\lambda,\mu}(f)$, valid at finite N and for symbols not in Szegő class. We show how several known and new results follow from this result in Section 3. We analyze our formulas specifically for the case of Szegő symbols in Section 4, exploiting their Wiener-Hopf factorization. In Section 5, we discuss the case of a Toeplitz minor when the symbol has FH singularities. Finally, in the last Section, we study the inverses of Toeplitz matrices. We combine our results for FH singularitites with the use of the Duduchava-Roch formula [17, 29, 5], to obtain a formula for the specialization of a class of skew-Schur polynomials.

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2. Finite N Toeplitz minors

In the next theorem we drop the integrability condition on f. We implicitly extend the definition of Toeplitz matrix generated by a function in $L^1(\mathbb{T})$ to functions having only formal series expansions. These can be understood as matrices generated by sequences $(d_k)_{k\in\mathbb{Z}}$. We will discuss later the regularity conditions on f, to see how our result translates in the context of Toeplitz operators.

Theorem 3. *Let $f(e^{i\theta}) = \sum_{k\in\mathbb{Z}} d_k e^{ik\theta}$ be a function on $\mathbb{T}$, and let λ, μ be partitions with $l(\lambda), l(\mu) \leq N$. Suppose there exist functions f^- and f^+ on $\mathbb{T}$ such that*

$$f^-(e^{i\theta}) = 1 + \sum_{k\leq -1} d_k^- e^{ik\theta}, \qquad f^+(e^{i\theta}) = 1 + \sum_{k\geq 1} d_k^+ e^{ik\theta}$$

and

$$f(e^{i\theta}) = f^-(e^{i\theta}) f^+(e^{i\theta}). \tag{2}$$

Then, the Toeplitz minor $D_N^{\lambda,\mu}(f)$ verifies

$$D_N^{\lambda,\mu}(f) = \sum_{l(\nu)\leq N} D_N^{\nu,\mu}(f^-) D_N^{\lambda,\nu}(f^+). \tag{3}$$

Proof. The main ingredients in the proof are the fact that the Toeplitz matrices generated by f^- and f^+ verify [7]

$$T(f) = T(f^-)T(f^+),$$

the Cauchy-Binet formula and the following lemma.

Lemma 1. *Let ν be a partition verifying $\nu \subset (K^N)$, and consider the partition $\overleftarrow{\nu}^K = (K - \nu_N, \ldots, K - \nu_1)$ which is obtained by rotating 180º the complement of ν in the diagram of the rectangular partition (K^N). Then the Schur polynomial s_ν verifies*

$$s_\nu(x_1^{-1}, \ldots, x_N^{-1}) = s_{\overleftarrow{\nu}^K}(x_1, \ldots, x_N) \prod_{j=1}^N x_j^{-K}.$$

A proof of the lemma follows performing elementary computations in the determinantal definition of the Schur polynomials.

If R, S are two strictly increasing sequences of natural numbers, we denote by $\det_{R,S} M$ the minor of the matrix M obtained by taking the rows and columns of M indexed by R and S, respectively. With this notation we have

$$D_N^{\lambda,\mu}(f) = \det_{R,S} T(f),$$

where the sequences R, S are given by

$$R = (r_j)_{j=1}^N = (j - \lambda_j + \max\{\lambda_1, \mu_1\})_{j=1}^N, \quad S = (s_k)_{k=1}^N = (k - \mu_k + \max\{\lambda_1, \mu_1\})_{k=1}^N.$$

Using theorem 1 and the lemma above with $K = \max\{\lambda_1, \mu_1\}$ we see that

$$D_N^{\lambda,\mu}(f) = \int_{U(N)} \overline{s_\lambda(M)} s_\mu(M) f(M) dM = \int_{U(N)} s_{\overleftarrow{\lambda}^K}(M) \overline{s_{\overleftarrow{\mu}^K}(M)} f(M) dM = \det_{R,S} T(f),$$

where $r_j = j + \mu_{N+1-j}$ and $s_k = k + \lambda_{N+1-k}$. Now, if we denote by $T_{N\times\infty}(f^-)$ and $T_{\infty\times N}(f^+)$ the matrices consisting of the first N rows of $T(f^-)$ and the first N columns of $T(f^+)$ respectively, we have

$$T_N(f) = T_{N\times\infty}(f^-) T_{\infty\times N}(f^+).$$

Since choosing a set of rows and columns to form a minor in a product of matrices AB amounts to choosing the same rows from A and columns from B and then taking the product of these submatrices, the Cauchy-Binet formula gives[2]

$$\det_{R,S} T(f) = \det_{R,S} T(f^-)T(f^+) = \sum_T \det_{R,T} T(f^-) \det_{T,S} T(f^+),$$

where the summation is over all the strictly increasing sequences $T = (t_1, \ldots, t_N)$ of length N of positive integers. There is a correspondence between such sequences and partitions ν of length $l(\nu) \leq N$, given by $\nu_{N+1-j} = t_j - j$, for $j = 1, ..., N$. Thus, for each T we have

$$\det_{T,S} T(f^+) = \det(d^+_{t_j - s_k})_{j,k=1}^N = \det(d^+_{j+\nu_{N+1-j}-(k+\lambda_{N+1-k})})_{j,k=1}^N = D_N^{\lambda,\nu}(f^+).$$

We have reversed the order of the rows and columns to establish the last identity above. Proceeding analogously with $\det_{R,T} T(f^-)$ we arrive at the desired conclusion. □

The main feature of the above formula is that it gives an expression for a Toeplitz minor valid for every N, whereas other works on Toeplitz minors [11, 13, 14, 25, 34] have mostly focused on the $N \to \infty$ limit of the minors. Moreover, several known and new results follow from this expression, as we show in the following.

3. Symmetric functions as symbols

We start by recalling some basic results involving symmetric functions (see [26, 31] for example). We denote $z = e^{i\theta}$ in the following, and treat z as a formal variable.

If $x = (x_1, x_2, ...)$ is a set of variables, the power-sum symmetric polynomials p_k are defined as $p_k(x) = x_1^k + x_2^k + \ldots$ for every $k \geq 1$, and $p_0(x) = 1$. They are related to the elementary symmetric polynomials $e_k(x)$ and the complete homogeneous polynomials $h_k(x)$ by the formulas

$$\begin{aligned} \exp\left(\sum_{k=1}^{\infty} \frac{p_k(x)}{k} z^k\right) &= \sum_{k=0}^{\infty} h_k(x) z^k = \prod_{j=1}^{\infty} (1 - x_j z)^{-1} = H(x; z), \\ \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{p_k(x)}{k} z^k\right) &= \sum_{k=0}^{\infty} e_k(x) z^k = \prod_{j=1}^{\infty} (1 + x_j z) = E(x; z). \end{aligned} \tag{4}$$

We also set $p_k(x) = h_k(x) = e_k(x) = 0$ for negative k. We will see H and E as functions on the unit circle $\mathbb{T}$, defined by the formal series above, and we will use indistinctly their infinite product expression. The classical Jacobi-Trudi identities express Schur polynomials as Toeplitz minors generated by the above functions

$$\begin{aligned} s_\mu(x) &= \det\left(h_{j-(k-\mu_k)}(x)\right)_{j,k=1}^N = D_N^{\varnothing,\mu}\left(H(x;z)\right), \\ s_{\mu'}(x) &= \det\left(e_{j-(k-\mu_k)}(x)\right)_{j,k=1}^N = D_N^{\varnothing,\mu}\left(E(x;z)\right), \end{aligned}$$

where $l(\mu), l(\mu') \leq N$, respectively. More generally, skew-Schur polynomials can be expressed as the minors

$$s_{\mu/\lambda}(x) = D_N^{\lambda,\mu}(H(x;z)), \qquad s_{(\mu/\lambda)'}(x) = D_N^{\lambda,\mu}(E(x;z)),$$

where $l(\mu), l(\mu') \leq N$ respectively. A skew-Schur polynomial vanishes if $\lambda \not\subseteq \mu$, which is consistent with the Toeplitz minor structure and the fact that the Toeplitz matrices above

[2] We are actually using the infinite dimensional generalization of the Cauchy-Binet formula that appears in [33].

are triangular. A central result in the theory of symmetric functions is the Cauchy identity, and its dual form

$$\sum_{\nu} s_\nu(x)s_\nu(y)=\prod_{j=1}^{\infty}\prod_{k=1}^{\infty}\frac{1}{1-x_jy_k},\qquad \sum_{\nu} s_\nu(x)s_{\nu'}(y)=\prod_{j=1}^{\infty}\prod_{k=1}^{\infty}(1+x_jy_k),$$

where $y=(y_1,y_2,\dots)$ is another set of variables and the sums run over all partitions ν.

3.1. **Gessel's theorem for minors.** Gessel [20, Thm. 16] obtained the following expression for the Toeplitz determinant generated by the symbol $f(z)=H(y;z^{-1})H(x;z)$

$$D_N\left(\prod_{k=1}^{\infty}(1-y_kz^{-1})^{-1}\prod_{j=1}^{\infty}(1-x_jz)^{-1}\right)=\sum_{l(\nu)\le N}s_\nu(y)s_\nu(x).$$

Comparing the right hand side with the sum in Cauchy identity and recalling that the Schur polynomial $s_\nu(x_1,...,x_N)$ vanishes if $l(\nu)>N$ one obtains a well known identity of Baxter [3, Lemma 7.4]

$$D_N\left(\prod_{k=1}^{K}\frac{1}{(1-y_kz^{-1})}\prod_{j=1}^{J}\frac{1}{(1-x_jz)}\right)=\prod_{j=1}^{J}\prod_{k=1}^{K}\frac{1}{1-x_jy_k},\tag{5}$$

valid when $J,K\le N$. The following identities, where one of the sets of variables is infinite and one of the factors H may be replaced by E, also hold for $J,K\le N$

$$\begin{aligned}D_N\left(H(y_1,...,y_K;z^{-1})H(x;z)\right)&=\prod_{j=1}^{\infty}\prod_{k=1}^{K}\frac{1}{1-x_jy_k},\\ D_N\left(H(y_1,...,y_K;z^{-1})E(x;z)\right)&=\prod_{j=1}^{\infty}\prod_{k=1}^{K}(1+x_jy_k),\\ D_N\left(E(y;z^{-1})H(x_1,...,x_J;z)\right)&=\prod_{j=1}^{J}\prod_{k=1}^{\infty}(1+x_jy_k).\end{aligned}\tag{6}$$

An important feature of the last four formulas is that the right hand side is independent of N. Note, however, that no such identity is available for Toeplitz determinants generated by symbols of the type $E(y;z^{-1})E(x;z)$. This will be relevant later. Using the Toeplitz minor expression of skew-Schur polynomials, formula (3) becomes

$$D_N^{\lambda,\mu}\left(\prod_{j=1}^{\infty}(1-y_jz^{-1})^{-1}\prod_{j=1}^{\infty}(1-x_jz)^{-1}\right)=\sum_{l(\nu)\le N}s_{\nu/\mu}(y)s_{\nu/\lambda}(x).\tag{7}$$

This gives a generalization of Gessel's theorem for minors of Toeplitz matrices. This identity was also obtained by Adler and van Moerbeke [1, Prop. 3.1], in its equivalent matrix integral form. The analogous formulas for Toeplitz minors generated by other symbols are

$$\begin{aligned}D_N^{\lambda,\mu}\left(\prod_{j=1}^{\infty}(1-y_jz^{-1})^{-1}\prod_{j=1}^{\infty}(1+x_jz)\right)&=\sum_{l(\nu)\le N}s_{\nu/\mu}(y)s_{(\nu/\lambda)'}(x),\\ D_N^{\lambda,\mu}\left(\prod_{j=1}^{\infty}(1+y_jz^{-1})\prod_{j=1}^{\infty}(1-x_jz)^{-1}\right)&=\sum_{l(\nu)\le N}s_{(\nu/\mu)'}(y)s_{\nu/\lambda}(x),\\ D_N^{\lambda,\mu}\left(\prod_{j=1}^{\infty}(1+y_jz^{-1})\prod_{j=1}^{\infty}(1+x_jz)\right)&=\sum_{l(\nu)\le N}s_{(\nu/\mu)'}(y)s_{(\nu/\lambda)'}(x).\end{aligned}\tag{8}$$

3.2. **Averages of Toeplitz minors.** We use in the following the term *Toeplitz average* to refer to the quotient of a Toeplitz minor and the Toeplitz determinant generated by the same symbol. We start by considering the $N \to \infty$ limit of a Toeplitz average generated by the symbol $f(z) = H(y; z^{-1})H(x; z)$. Using the Cauchy identity and [26, Ex. 5.26] we obtain

$$\lim_{N\to\infty}\left[\frac{D_N^{\lambda,\mu}\left(\prod_{j=1}^{\infty}(1-y_jz^{-1})^{-1}\prod_{j=1}^{\infty}(1-x_jz)^{-1}\right)}{D_N\left(\prod_{j=1}^{\infty}(1-y_jz^{-1})^{-1}\prod_{j=1}^{\infty}(1-x_jz)^{-1}\right)}\right] = \sum_{\nu} s_{\lambda/\nu}(y)s_{\mu/\nu}(x). \tag{9}$$

This sum is considerably simpler than the one in equation (7), as it actually runs over the partitions ν verifying $\nu \subset \lambda, \mu$. Note also that if λ is the empty partition the right hand side simplifies to $s_\mu(x)$. For Toeplitz averages generated by other symbols, the dual Cauchy identity and [26, Ex. 5.26] give the analogous expressions

$$\begin{aligned}
\lim_{N\to\infty}\left[\frac{D_N^{\lambda,\mu}\left(H(y;z^{-1})E(x;z)\right)}{D_N\left(H(y;z^{-1})E(x;z)\right)}\right] &= \sum_{\nu} s_{\lambda/\nu}(y)s_{(\mu/\nu)'}(x),\\
\lim_{N\to\infty}\left[\frac{D_N^{\lambda,\mu}\left(E(y;z^{-1})H(y;z)\right)}{D_N\left(E(y;z^{-1})H(y;z)\right)}\right] &= \sum_{\nu} s_{(\lambda/\nu)'}(y)s_{\mu/\nu}(x),\\
\lim_{N\to\infty}\left[\frac{D_N^{\lambda,\mu}\left(E(y;z^{-1})E(x;z)\right)}{D_N\left(E(y;z^{-1})E(x;z)\right)}\right] &= \sum_{\nu} s_{(\lambda/\nu)'}(y)s_{(\mu/\nu)'}(x).
\end{aligned} \tag{10}$$

Some of these asymptotic formulas also hold for finite N. Indeed, if $J, K \le N$, the use of the Baxter-type identities (6) in [26, Ex. 5.26] gives

$$\begin{aligned}
D_N^{\lambda,\mu}\left(\prod_{k=1}^{K}(1-y_kz^{-1})^{-1}\prod_{j=1}^{\infty}(1-x_jz)^{-1}\right) &= \prod_{j=1}^{\infty}\prod_{k=1}^{K}\frac{1}{1-x_jy_k}\sum_{\nu} s_{\lambda/\nu}(y_1,...,y_K)s_{\mu/\nu}(x),\\
D_N^{\lambda,\mu}\left(\prod_{k=1}^{K}(1-y_kz^{-1})^{-1}\prod_{j=1}^{\infty}(1+x_jz)\right) &= \prod_{j=1}^{\infty}\prod_{k=1}^{K}(1+x_jy_k)\sum_{\nu} s_{\lambda/\nu}(y_1,...,y_K)s_{(\mu/\nu)'}(x),\\
D_N^{\lambda,\mu}\left(\prod_{k=1}^{\infty}(1+y_kz^{-1})\prod_{j=1}^{J}(1-x_jz)^{-1}\right) &= \prod_{j=1}^{J}\prod_{k=1}^{\infty}(1+x_jy_k)\sum_{\nu} s_{(\lambda/\nu)'}(y)s_{\mu/\nu}(x_1,...,x_J).
\end{aligned} \tag{11}$$

We emphasize that the double product in the right hand side of the identities above is precisely the Toeplitz determinant generated by the corresponding symbol, appearing in the left hand side. This means that we can express the minors in the left hand side as a product of the corresponding determinant and an additional term, which is also independent of N, as long as N is large enough. However, no such factorization is available for symbols of the type $E(y; z^{-1})E(x; z)$. This is precisely because of the lack of an identity of the type (5),(6) for the Toeplitz determinants generated by these symbols, as alluded before. Nevertheless, a different formula holds for these determinants and for minors.

3.3. **The $E(y; z^{-1})E(x; z)$ case.** The expression

$$s_{(N^K)}(x) = \det\left(e_{j-k+K}(x)\right)_{j,k=1}^{N} = D_N\left(\prod_{j=1}^{K}x_j\prod_{j=1}^{K}(1+x_j^{-1}z^{-1})\prod_{j=K+1}^{\infty}(1+x_jz)\right)$$

is well known (see for example [12]). The Jacobi-Trudi identity for skew-Schur polynomials readily gives the following

$$s_{(K^N+\mu/\lambda)'}(y_1,\dots,y_K,x) = D_N^{\lambda,\mu}\left(\prod_{k=1}^K y_k \prod_{k=1}^K (1+y_k^{-1}z^{-1}) \prod_{j=1}^\infty (1+x_j z)\right),$$

after a relabelling of the variables. Hence, comparing with the last identity in (8), we obtain

$$\begin{aligned} D_N^{\lambda,\mu}\left(E(y_1,...,y_K;z^{-1})E(x;z)\right) &= \sum_{\nu\subset(K^N)} s_{(\nu/\mu)'}(y_1,\dots,y_K)s_{(\nu/\lambda)'}(x) = \\ &= \prod_{k=1}^K y_k^N s_{(K^N+\mu/\lambda)'}(y_1^{-1},\dots,y_K^{-1},x). \end{aligned} \tag{12}$$

This gives an expression for Toeplitz minors generated by symbols of the type $E(y;z^{-1})E(x;z)$, valid for finite N and simpler than the sum (8). The second equality establishes an identity between symmetric functions (see [10, Lemma 2] for an analogous result, in the case $\lambda=\mu=\varnothing$).

As an application of the results in this Section, we give simple proofs of two classical results of Baxter and Schmidt.

Corollary 1 (Thm. 1 in [4]). *Let $f(z)=1+\sum_{k\geq 1} d_k z^k$ be a formal power series. For every $N,K\geq 1$ we have*

$$D_N(z^{-K}f(z)) = (-1)^{NK} D_K(z^{-N}f^{-1}(z)) \tag{13}$$

Proof. Assume, without loss of generality, that $N\geq K$. Since f is a formal series, we can express it as $f(z)=H(x;z)$, for some x (recall that the functions h_k appearing in (4) are algebraically independent). We use the following property, immediate from the formulas (7),(8)

$$D_N^{\lambda,\mu}(f) = (-1)^{|\lambda|+|\mu|} D_N^{\lambda',\mu'}(f^{-1}).$$

Since the determinant in the left hand side of (13) has the minor expression $D_N(z^{-K}f(z)) = D_N^{\varnothing,(K^N)}(f)$, the above property gives

$$D_N^{\varnothing,(K^N)}(f) = (-1)^{NK} D_N^{\varnothing,(N^K)}(f^{-1}) = (-1)^{NK} D_K^{\varnothing,(N^K)}(f^{-1}).$$

We have used in the last equality the fact that $D_N(g)=D_K(g)$ for every $N\geq K$ if the function g is of the form $g(z)=z^{-K}H(x;z)$ (last equation in (6)). Using again the minor expression $D_K^{\varnothing,(N^K)}(f^{-1}) = D_K(z^{-N}f^{-1}(z))$ we arrive at the desired conclusion. □

Corollary 2 (Thm. 3 in [4]). *The Toeplitz determinant generated by the symbol*

$$f(z) = z^{-K} + d_1 z^{-K+1} + \cdots + d_K + d_{K+1}z + \cdots + d_{K+J}z^J$$

verifies

$$D_N(f) = (-1)^{NK}\sum_I \left(\prod_{i\in I} x_i^{N+J}\right)\left(\prod_{i\in I}\prod_{j\in I^c}\frac{1}{x_i-x_j}\right),$$

where the sum runs over all the subsets I of cardinality K of $\{1,2,...,K+J\}$, I^c denotes the complement of I in the set $\{1,2,...,K+J\}$, and the x_j are the roots of the polynomial

$$z^{K+J} + d_1 z^{K+J-1} + \cdots + d_{K+J-1}z + d_{K+J}.$$

Proof. This is precisely the case $\lambda=\mu=\varnothing$ of formula (12), after the appropriate relabelling of the variables and the use of Cauchy identity. □

The specific setting where the symbol generating the Toeplitz matrix admits a Wiener-Hopf factorization is presented in the following Section.

4. Szegő symbols

We first discuss the regularity of the symbol f. If a function f is continuous and non-zero on $\mathbb{T}$, has winding number zero, and belongs to the Wiener algebra, it has a Wiener-Hopf factorization [7]. In particular, for a function f satisfying the decay conditions in the strong Szegő limit theorem (namely $f(e^{i\theta}) = \exp(\sum_{k\in\mathbb{Z}} c_k e^{ik\theta})$ with $\sum_{k\in\mathbb{Z}} |c_k| < \infty$ and $\sum_{k\in\mathbb{Z}} |k||c_k|^2 < \infty$), these conditions are satisfied, and its Wiener-Hopf factorization is given by

$$f(e^{i\theta})e^{-c_0} = \exp\left(\sum_{k\leq -1} c_k e^{ik\theta}\right) \exp\left(\sum_{k\geq 1} c_k e^{ik\theta}\right),$$

which is precisely of the form (2). Hence, we can express each of the two factors above as a specialization of one of the functions appearing in equation (4), since both the e_k and the h_k are algebraically independent families in the ring of symmetric functions. That is, there exist some $\tilde{x}$,$\tilde{y}$, such that the function f is of the form

$$f(z) = f^-(\tilde{y}; z^{-1})f^+(\tilde{x}; z),$$

where each of the factors f^- and f^+ is of the type H or E defined in (4). The precise factorization depends on the number of nonzero Fourier coefficients of the factors: if f^+ is a polynomial of degree J, then the specialization is

$$f^+(z) = 1 + d_1 z + \cdots + d_J z^J = E(\tilde{x}_1, \ldots, \tilde{x}_J; z),$$

where the $\tilde{x}$ are the roots of the polynomial

$$\prod_{j=1}^{J}(z - \tilde{x}_j) = \sum_{k=0}^{J}(-1)^{J+k} e_{J-k}(\tilde{x}) z^k = z^J - d_1^+ z^{J-1} + \cdots + (-1)^J d_J^+.$$

If f^+ has an infinite number of nonzero Fourier coefficients, then both the specializations $f^+(z) = E(\tilde{x}; z)$ and $f^+(z) = H(\tilde{x}; z)$ are available. An analogous reasoning holds for f^- and the variables y.

Therefore, all the results in the previous Section reproduce the setting where the symbol f satisfies the decay conditions of the strong Szegő limit theorem, simply by specializing the variables in any of the formulas to the $\tilde{x}$,$\tilde{y}$. In particular, the constant appearing in the right hand side of the strong Szegő limit theorem

$$\exp\left(\sum_{k=1}^{\infty} k c_k c_{-k}\right)$$

is just the double product appearing in the Cauchy identity (or its dual), for this choice of variables. Also, the double sum appearing in Theorem 2 of Bump and Diaconis

$$\sum_{\phi\vdash n}\sum_{\psi\vdash m} \chi_\phi^\lambda\, \chi_\psi^\mu\, z_\phi^{-1} z_\psi^{-1} \Delta(f, \phi, \psi),$$

has an equivalent expression as the right hand side of equations (9),(10), with the specific form depending on the choice of factorization. Moreover, we can extend Theorem 2 to finite N for the class of symbols appearing in equations (6).

Theorem 4. *Let $f(e^{i\theta}) = \exp(\sum_{k\in\mathbb{Z}} c_k e^{ik\theta})$, and suppose there exist $\tilde{y}_1, ..., \tilde{y}_K$ with $|\tilde{y}_k| < 1$ for $1 \leq k \leq K$ such that*

$$\tilde{y}_1^k + \tilde{y}_2^k + \cdots + \tilde{y}_K^k = kc_{-k}$$

for every $k \geq 1$ or

$$\tilde{y}_1^k + \tilde{y}_2^k + \cdots + \tilde{y}_K^k = kc_k$$

for every $k \geq 1$. Then, for any pair of partitions λ, μ, and any N such that $N \geq K, l(\lambda), l(\mu)$ we have

$$D_N^{\lambda,\mu}(f) = D_N(f) \sum_{\phi\vdash n} \sum_{\psi\vdash m} \chi_\phi^\lambda \, \chi_\psi^\mu \, z_\phi^{-1} z_\psi^{-1} \Delta(f, \phi, \psi),$$

where n and m are the weights of λ and μ respectively and the sum is as in theorem 2.

The proof follows from noting that the Fourier coefficients c_k, c_{-k} are given by the power sums $p_k(\tilde{x}), p_k(\tilde{y})$, for every $k \geq 1$. These coefficients can be obtained recursively from the Fourier coefficients of the symbol, by means of Newton's identities [26]

$$kh_k(x) = \sum_{j=1}^{k} p_j(x) h_{k-j}(x), \qquad ke_k(x) = \sum_{j=1}^{k} (-1)^{j-1} p_j(x) e_{k-j}(x) \qquad (k \geq 1).$$

In the notation of Section 2, these equations relate the Fourier coefficients d_k^+ and d_k^- of the factors f^+ and f^- with the Fourier coefficients c_k and c_{-k} of their exponentials (for $k \geq 1$).

We briefly discuss now the role of the specialization of the variables x,y in this Section. Recall that all of the formulas in the previous Section hold as formal series identities in the variables x,y. Introducing the regularity of the symbol f (via the specializations $x \to \tilde{x}$, $y \to \tilde{y}$) amounts to imposing certain decay conditions on the families of symmetric functions involved, in order to make the objects appearing in the formulas well-defined (i.e. the functions H, E, the Toeplitz determinants generated by them, and so on). At the level of the $\tilde{x}, \tilde{y}$, it suffices to consider absolutely convergent sequences $\tilde{x} = (\tilde{x}_j)$ and $\tilde{y} = (\tilde{y}_j)$ of points in the interior of the unit disk to achieve this regularity. In particular, the decay conditions in the strong Szegő limit theorem are verified in these conditions.

We emphasize that despite working with functions with a Wiener-Hopf factorization may seem more natural from the point of view of Toeplitz operators, these can be realized as a particular case in the more general setting of symmetric functions. This allows to obtain results for functions that violate the regularity conditions in the strong Szegő limit theorem, as we show in what follows.

5. Fisher-Hartwig symbols.

We now discuss Fisher-Hartwig (or FH) singularities [19, 7] as an example of symbols which do not verify the decay conditions in the strong Szegő limit theorem. These symbols can also be seen as specializations of the results in Section 3.

5.1. The pure Fisher-Hartwig singularity.

The so-called pure FH singularity is the symbol defined on $\mathbb{T}$ by the expression

$$|1 - e^{i\theta}|^{2\alpha} e^{i\beta(\theta-\pi)} \qquad (0 < \theta < 2\pi), \tag{14}$$

where $\mathrm{Re}(\alpha) > -1/2$ and $\beta \in \mathbb{C}$. The factor $|1 - e^{i\theta}|^{2\alpha}$ may have a zero, a pole, or an oscillatory singularity, while the factor $e^{i\beta(\theta-\pi)}$ has a jump if β is not an integer. Thus, depending on the different values of the parameters α and β, the symbol above may violate the conditions required for a symbol to have a Wiener-Hopf factorization. It will be more convenient to work with the equivalent factorization [7]

$$\varphi_{\gamma,\delta}(e^{i\theta}) = (1-e^{i\theta})^{\gamma}(1-e^{-i\theta})^{\delta}.$$

This function coincides with (14) if $\gamma = \alpha+\beta$ and $\delta = \alpha-\beta$, and it can be expressed as the exponential of the function $\gamma\log(1-e^{i\theta}) + \delta\log(1-e^{-i\theta})$, whose Fourier coefficients

$$c_k = \gamma\frac{(-1)^k}{k}, \qquad c_{-k} = \delta\frac{(-1)^k}{k} \qquad (k\geq 1),$$

do not satisfy the decay conditions in the strong Szegő limit theorem.

We will assume in the following that γ and δ are positive integers. While some of the formulas are valid in the more general conditions $\mathrm{Re}(\gamma+\delta) > -1$ (in particular, (15) and (16)), an expression of the function $\varphi_{\gamma,\delta}$ as a product of symmetric functions is only available when γ and δ are positive integers. Indeed, in these conditions, we can write $\varphi_{\gamma,\delta}$ as the specialization

$$\varphi_{\gamma,\delta}(e^{i\theta}) = E(\underbrace{1,...,1}_{\delta};-e^{-i\theta})E(\underbrace{1,...,1}_{\gamma};-e^{i\theta}),$$

where E is given by (4). Note that we can drop the sign before the $e^{i\theta}, e^{-i\theta}$ in the right hand side above, since the Toeplitz determinant generated by a function $f(z)$ coincides with that generated by $f(az)$, for any a.

Böttcher and Silbermann [8] obtained an explicit expression for the Toeplitz determinant generated by the pure FH singularity, that reads

$$D_N(\varphi_{\gamma,\delta}) = G(N+1)\frac{G(\gamma+\delta+N+1)}{G(\gamma+\delta+1)}\frac{G(\gamma+1)}{G(\gamma+N+1)}\frac{G(\delta+1)}{G(\delta+N+1)}, \tag{15}$$

where G is the Barnes function [2]. It follows from the well known asymptotics of the Barnes G function that this determinant has the asymptotic behaviour

$$D_N(\varphi_{\gamma,\delta}) \overset{N\to\infty}{\sim} N^{\gamma\delta}\frac{G(\gamma+1)G(\delta+1)}{G(\gamma+\delta+1)}.$$

An explicit expression for a Toeplitz minor generated by the pure FH singularity indexed by a single partition μ is[3]

$$D_N^{\varnothing,\mu}(\varphi_{\gamma,\delta}) = G(N+1)\frac{G(\gamma+\delta+N+1)}{G(\gamma+\delta+1)}s_\mu(\underbrace{1,...,1}_{N})\prod_{l=1}^{N}\frac{1}{\Gamma(\gamma-\mu_l+l)}\frac{1}{\Gamma(\delta+\mu_l+N-l+1)}, \tag{16}$$

whenever $\mu_1 \leq \gamma$ (the minor vanishes otherwise). If $l(\mu) = M \leq N$ for a fixed M, some computation leads to the equivalent formula

$$D_N^{\varnothing,\mu}(\varphi_{\gamma,\delta}) = D_N(\varphi_{\gamma,\delta})\frac{G(N-M+1)}{G(N+1)}\frac{G(\delta+N+1)}{G(\delta+N-M+1)}s_{\mu'}(\underbrace{1,...,1}_{\gamma})\prod_{k=1}^{M}\frac{(\mu_k+N-k)!}{(\delta+\mu_k+N-k)!}.$$

From this expression we obtain the asymptotics

$$D_N^{\varnothing,\mu}(\varphi_{\gamma,\delta}) \overset{N\to\infty}{\sim} N^{\gamma\delta}\frac{G(\gamma+1)G(\delta+1)}{G(\gamma+\delta+1)}s_{\mu'}(\underbrace{1,...,1}_{\gamma}).$$

Thus, comparing the $N\to\infty$ limit of a Toeplitz minor and the Toeplitz determinant generated by $\varphi_{\gamma,\delta}$ we recover the right hand side of equation (10), as expected. As mentioned before, this is a consequence of the fact that the behaviour of Toeplitz averages is the same for both symbols in the Szegő class and for those with FH singularities (and in particular, Theorem 2 also holds

[3] See the appendix for a proof.

for these symbols). We mention that the expression (16) has been obtained before in the case $\gamma = \delta$, see [10], for instance.

Specializing the identity (12) we obtain the formula

$$D_N^{\lambda,\mu}(\varphi_{\gamma,\delta}) = s_{(\delta^N+\mu/\lambda)'}(\underbrace{1,...,1}_{\gamma+\delta}). \tag{17}$$

If λ is the empty partition, one recovers from this formula the expression (16), using the well know identity

$$s_\mu(\underbrace{1,...,1}_{N}) = \frac{1}{G(N+1)} \prod_{1\le j<k\le N} (\mu_j - \mu_k + k - j), \tag{18}$$

valid for any $N \ge l(\mu)$. The specialization of skew-Schur polynomials will be discussed in the next Section.

Using theorem 1 and the factorization (14), the Toeplitz minor generated by the pure FH singularity $D_N^{\lambda,\mu}(\varphi_{\gamma,\delta})$ can be written in the equivalent integral form

$$\frac{1}{N!}\frac{1}{(2\pi)^N}\int_0^{2\pi}...\int_0^{2\pi} s_\lambda(e^{-i\theta})s_\mu(e^{i\theta})\prod_{j=1}^N e^{\frac{1}{2}i\theta_j(\gamma-\delta)}|1+e^{i\theta_j}|^{\gamma+\delta}\prod_{1\le j<k\le N}|e^{i\theta_j}-e^{i\theta_k}|^2 d\theta_1...d\theta_N, \tag{19}$$

where we denote $s_\mu(e^{i\theta}) = s_\mu(e^{i\theta_1},\dots,e^{i\theta_N})$. This integral, without the factor $1/N!$ and the Schur polynomial insertions, is a particular case of the unitary version of Selberg integral known as Morris integral. Explicit formulas are known [18] for the evaluation of the Morris integral, with and without the insertion of a single Schur polynomial s_μ . These correspond to the Toeplitz minor $N!D_N^{\varnothing,\mu}(\varphi_{\gamma,\delta})$ and the Toeplitz determinant $N!D_N(\varphi_{\gamma,\delta})$ respectively, and coincide with formulas (16) and (15). The expression of the Morris integral as the specialization of a single Schur polynomial, given by (17), appears to be new. Moreover, this formula corresponds to the insertion of two Schur polynomials in the integral, as shown in equation (19). Results for (19) have been obtained for the specific case $\gamma = -\delta$ [23, Th. 2].

At the end of Section 6, we will obtain a formula for the specialization of skew-Schur polynomials, which gives an explicit evaluation of (19) when the Schur polynomials in (19) reduce to elementary symmetric polynomials.

5.2. **The general Fisher-Hartwig symbol.** The general expression of a symbol with FH singularities is [19, 7]

$$g(e^{i\theta}) = f(e^{i\theta})\prod_{r=1}^R \varphi_{\gamma_r,\delta_r}(e^{i\theta}/e^{i\theta_r}),$$

where f is a symbol verifying the decay conditions in the strong Szegő limit theorem and the $e^{i\theta_r}$ are pairwise distinct points in $\mathbb{T}$. Thus, the symbol g is the product of a symbol in the Szegő class and a finite number of pure FH singularities. Since f has a Wiener-Hopf factorization, we can write it as $f(e^{i\theta}) = E(\tilde{y}; e^{-i\theta})E(\tilde{x}; e^{i\theta})$ for some $\tilde{x}, \tilde{y}$, as discussed in Section 4. On the other hand, each of the pure singularities in the product can be factored as

$$\varphi_{\gamma_r,\delta_r}(e^{i\theta}/e^{i\theta_r}) = E(\underbrace{-e^{i\theta_r},...,-e_r^{i\theta}}_{\delta_r}; e^{-i\theta})E(\underbrace{-e_r^{-i\theta},...,-e_r^{-i\theta}}_{\gamma_r}, e^{i\theta}).$$

Thus, the formulas obtained in Section 3 for the $E(y, z^{-1})E(x; z^{-1})$ case apply to this class of symbols, after the specialization of the variables x, y. As an example, we write down the expression for the large-N limit of the Toeplitz average generated by g

$$\lim_{N\to\infty}\left[\frac{D_N^{\lambda;\mu}(g)}{D_N(g)}\right] =$$

$$\sum_{\nu} s_{(\lambda/\nu)'}(\underbrace{-e^{i\theta_1},...,-e^{i\theta}_1}_{\delta_1},...,\underbrace{-e^{i\theta_R},...,-e^{i\theta}_R}_{\delta_R},\tilde{y})\, s_{(\mu/\nu)'}(\underbrace{-e^{-i\theta_1},...,-e^{-i\theta}_1}_{\gamma_1},...,\underbrace{-e^{-i\theta_R},...,-e^{-i\theta}_R}_{\gamma_R},\tilde{x}).$$

6. Inverses of Toeplitz matrices and skew-Schur specialization

The usual formula for the inversion of a matrix in terms of its cofactors reads, for the case of Toeplitz matrices, as follows

$$\left(T_N^{-1}(f)\right)_{j,k} = \left((-1)^{j+k}\frac{D_{N-1}^{(1^{k-1}),(1^{j-1})}(f)}{D_N(f)}\right)_{j,k},$$

where above and in the following, (1^M) denotes the partition consisting of M ones, and we assume the invertibility of the matrices involved. Hence, we see that the entries of the inverse of a Toeplitz matrix are (almost) Toeplitz averages. We show now how this applies to some of the symbols studied above.

Let f be a symbol of the form $f(z) = H(y_1,...,y_K;z^{-1})H(x;z)$, and let $N-1 \geq K$. Then, according to formula (11), the inverse of the Toeplitz matrix generated by f is

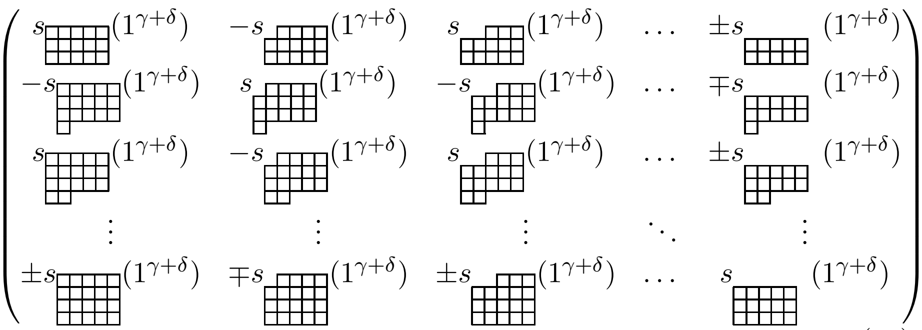
$$T_N^{-1}(H(y_1,...,y_K;z^{-1})H(x;z)) =$$

$$\begin{pmatrix} 1 & -e_1(y) & e_2(y) & \cdots \\ -e_1(x) & 1+e_1(x)e_1(y) & -(e_1(y)+e_1(x)e_2(y)) & \cdots \\ e_2(x) & -(e_1(x)+e_2(x)e_1(y)) & 1+e_1(x)e_1(y)+e_2(x)e_2(y) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (20)$$

Analogous expressions hold for the inverses of the matrices generated by the symbols appearing in equations (6), replacing the symmetric polynomials e_k by the h_k.

If f is of the form $f(z) = E(y;z^{-1})E(x;z)$, the sum of symmetric functions appearing in the above matrix is replaced by the skew-Schur polynomial (12) over the Toeplitz determinant generated by f. For instance, if f is the pure FH singularity $\varphi_{\gamma,\delta}$ we have

$$T_N^{-1}(\varphi_{\gamma,\delta}) = \frac{1}{D_N(\varphi_{\gamma,\delta})}$$



(21)

where the diagram indexing the Schur polynomial in the first entry of the matrix corresponds to the partition $(N-1)^\delta$, and we remove a box from its first row or add a box to the last (empty) row as we move to the right or downwards in the entries of the matrix, respectively. The signs of the last row and column should be read as $\pm = (-1)^{N+1}$ and $\mp = (-1)^N$.

Several results can be deduced from the explicit form of the above matrices, (20) and (21). We note the following:

Corollary 3. *Let f be a function on $\mathbb{T}$. The inverses of the Toeplitz matrices $T_N(f)$ are tridiagonal for every N if and only if the symbol f is of the form*[4]

$$f(z)=\frac{1}{1-x_1z}\frac{1}{1-y_1z^{-1}}, \tag{22}$$

for some x_1, y_1 (as above, this is to be understood as a formal result, and we impose the restriction $|\tilde{x}_1|, |\tilde{y}_1|<1$ when specializing).

Proof. If the symbol f only has a finite number of nonzero Fourier coefficients, then f is of the form $E(y_1,...,y_K;z^{-1})E(x_1,...,x_J;z)$, and the inverses of the associated Toeplitz matrices are clearly not tridiagonal. If f has an infinite number of nonzero Fourier coefficients, then f can be expressed as a specialization of one of the symbols appearing in equations (6) (possibly with an infinite number of variables). It follows from the form of the matrix (20) and the analogous ones for the other symbols, that for each of the factors of f, the conditions $e_k=0$, or $h_k=0$, for all $k\geq 2$, must be satisfied (if the factor is of the form H or E, respectively). But this means that f is of the form (22). □

The Duduchava-Roch formula [17, 29, 5] gives a relation between the Toeplitz matrix generated by the pure FH singularity and the matrices generated by each of the factors of $\varphi_{\gamma,\delta}$

$$T((1-z)^{\gamma})M_{\gamma+\delta}T((1-z)^{\delta})=\frac{\Gamma(\gamma+1)\Gamma(\delta+1)}{\Gamma(\gamma+\delta+1)}M_{\delta}T(\varphi_{\gamma,\delta})M_{\gamma},$$

where M_a is the diagonal matrix with entries $(M_a)_{k,k}=\binom{a+k-1}{k-1}$, for $k\geq 1$. Using the finite section version of this formula, one obtains the following expression for the (j,k)-th entry of the matrix $T_N^{-1}(\varphi_{\gamma,\delta})$ [5, Eq. 20]

$$(-1)^{j+k}\frac{(\gamma+j-1)!(\delta+k-1)!}{(j-1)!(k-1)!}\sum_{l=\max(j,k)}^{N}\frac{(l-1)!}{(\gamma+\delta+l-1)!}\binom{\gamma+l-k-1}{l-k}\binom{\delta+l-j-1}{l-j}.$$

The value of this formula for the corners of the matrix coincides with those of the matrix (21), which can be computed explicitly using the explicit expression for the determinant (15). Moreover, when $\delta=2$, the diagram indexing the Schur polynomial in the $(1,N-1)$-th entry of the matrix is an inverted hook. An expression for the specialization of such skew-Schur polynomials has been recently computed [27, Remark 8.4], and coincides with the value given by the Duduchava-Roch formula. By comparing the formula above with the rest of the entries of the matrix, we obtain the expression

$$s_{(\underbrace{N,\dots,N}_{d},j)/(k)}(1^M)=G(N+2)\frac{G(M+N+2)}{G(M+1)}\frac{G(M-d+1)}{G(M-d+N+2)}\frac{G(d+1)}{G(d+N+2)}\times$$
$$\times\frac{(M-d+j)!}{j!}\frac{(d+k)!}{k!}\sum_{l=\max(j,k)}^{N}\frac{l!}{(M+l)!}\binom{M-d+l-k-1}{l-k}\binom{d+l-j-1}{l-j},$$

valid for $j,k\leq N$ and $M>d$ (or $M\geq d$, if $j=0$). This gives a seemingly novel expression for the specialization of skew-Schur polynomials, indexed by partitions of the above form. Moreover, this also gives an explicit expression for the integral (19) whenever the Schur polynomials in the integrand reduce to elementary symmetric polynomials.

[4]The Toeplitz matrix generated by this symbol is an asymmetric version of the Kac-Murdock-Szegő matrix, whose explicit inverse is well-known, see [16] for instance.

Appendix: The case of a pure FH singularity: computation of the minor.

What follows is a proof of equation (16), which gives an explicit expression for a Toeplitz minor generated by the pure Fisher-Hartwig singularity $\varphi_{\gamma,\delta}$. We compute the minor $D_N^{\varnothing,\mu}(\varphi_{\gamma,\delta})$, where μ verifies $l(\mu) \leq N$. We also assume the condition $\mu_1 \leq \gamma$, since otherwise the minor vanishes. We follow the second of the two proofs given in [9] for the expression of the corresponding Toeplitz determinant.

The only non-zero Fourier coefficients of $\varphi_{\gamma,\delta}$ are

$$d_k = \frac{\Gamma(\gamma+\delta+1)}{\Gamma(\gamma-k+1)\Gamma(\delta+k+1)} \qquad (-\delta \leq k \leq \gamma).$$

This holds for any γ, δ such that $\mathrm{Re}(\gamma+\delta) > -1$ [7]. Thus, after taking out the factor $\Gamma(\gamma+\delta+1)^N$, and the products

$$\prod_{j=1}^{N} \frac{1}{\Gamma(\gamma-\mu_N+N-j+1)}, \qquad \prod_{k=1}^{N} \frac{1}{\Gamma(\delta+\mu_k+N-k+1)}$$

from the rows and columns of $D_N^{\varnothing,\mu}(\varphi_{\gamma,\delta})$ respectively, we obtain the determinant

$$\begin{vmatrix} \frac{\Gamma(\gamma-\mu_N+N)}{\Gamma(\gamma-\mu_1+1)}\frac{\Gamma(\delta+\mu_1+N)}{\Gamma(\delta+\mu_1+1)} & \frac{\Gamma(\gamma-\mu_N+N)}{\Gamma(\gamma-\mu_2+2)}\frac{\Gamma(\delta+\mu_2+N-1)}{\Gamma(\delta+\mu_2)} & \cdots & \frac{\Gamma(\delta+\mu_N+1)}{\Gamma(\delta+\mu_N-N+2)} \\ \frac{\Gamma(\gamma-\mu_N+N-1)}{\Gamma(\gamma-\mu_1)}\frac{\Gamma(\delta+\mu_1+N)}{\Gamma(\delta+\mu_1+2)} & \frac{\Gamma(\gamma-\mu_N+N-1)}{\Gamma(\gamma-\mu_2+1)}\frac{\Gamma(\delta+\mu_2+N-1)}{\Gamma(\delta+\mu_2+1)} & \cdots & \frac{\Gamma(\delta+\mu_N+1)}{\Gamma(\delta+\mu_N-N+3)} \\ \vdots & \vdots & & \vdots \\ \frac{\Gamma(\gamma-\mu_N+1)}{\Gamma(\gamma-\mu_1-N+2)} & \frac{\Gamma(\gamma-\mu_N+1)}{\Gamma(\gamma-\mu_1-N+3)} & \cdots & 1 \end{vmatrix}. \tag{23}$$

Subtracting $(\delta+\mu_N-N+1+j)$ times the $(j+1)$-th row from the j-th row, for $j = 1, ..., N-1$ we can make the last column vanish except for the 1 at the bottom, thus obtaining a determinant of order $N-1$. After extracting the factor

$$\prod_{k=1}^{N-1} (\gamma+\delta+1)(\mu_k-\mu_N+N-k)$$

from the columns of the matrix, and

$$\prod_{j=1}^{N-1} \frac{\Gamma(\gamma-\mu_N+j)}{\Gamma(\gamma-\mu_{N-1}+j)}$$

from the rows, we obtain a determinant with the same structure as (23), but with the following changes: N is replaced by $N-1$, δ is replaced by $\delta+1$ and μ is replaced by the partition $(\mu_1, \ldots, \mu_{N-1})$, that results from discarding the last part of μ. Making use of this recursive structure and formula (18) one obtains the expression (16) for the Toeplitz minor $D_N^{\varnothing,\mu}(\varphi_{\gamma,\delta})$, as desired.

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