Random matrices in gauge theory

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  - Formalism for exact solutions

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  - Examples of gauge theories with random matrix description
  - Example I: Pure Chern-Simons theory
  - Example II: Chern-Simons theory with susy matter
- Summary

- Let $H = (H_{jk})_{j,k=1}^{N}$ be a square $N \times N$ matrix with randomly distributed elements H_{jk}.This is a random matrix with respect to a probability distribution, defined by

$$P_{\beta}^{N}(H)dH = C \exp\left(-\beta \mathrm{Tr} V(H)\right) dH.$$

- The first and most studied ensembles are the Gaussian ensembles, $V(H) = H^2$. It can be shown that the previous expression is automatically restricted to the form

$$P(H) = \exp(-a\mathrm{Tr}H^2 + b\mathrm{Tr}H + c), \qquad a > 0.$$

if one postulates statistical independence of the matrix elements H_{jk}. There are three different ensembles defined, depending on the values of the parameter $\beta = 1, 2$ or 4.

- Ensembles of random $N \times N$ matrices H are defined by the following demands:

1. The probability $P(H)d[H]$ is invariant under any transformation $H \to U^{-1}HU$, where U is either an orthogonal $(\beta = 1)$, unitary $(\beta = 2)$ or symplectic $(\beta = 4)$ matrix. That is to say, if $H' = U^{-1}HU$ where U belongs to the unitary group $U(N;\beta)$, then $P(H')\,d[H'] = P(H)\,d[H]$.
2. The matrix elements which are not related by the symmetry of the matrix are statistically independent (Gaussian ensembles)

- Diagonalization: for each matrix H there is a matrix U that maps it onto its eigenvalues. The Jacobian of the transformation is $J_{\beta}(\{x_i\}) = \prod_{i<j} |x_i - x_j|^{\beta}$. The resulting expression is

$$P(x_1, ..., x_N) = C_N \prod_{i<j} |x_i - x_j|^{\beta} \exp[-\sum_{i=1}^{N} V(x_i)],$$

- The main relevant quantities are m-partial integrations over the previous N-dimensional probability density function.
- The simplest case to treat analytically is that of a Hermitian $(\beta = 2)$ ensemble.

- Let $p_N(x) = a_N x^N + \ldots$ be the *Nth* orthogonal polynomial associated to $\mathrm{e}^{-V(x)}$. k-point correlation function can be computed from the two-point kernel as follows (for $\beta = 2$)

$$R_k(x_1, x_2, \ldots, x_k) = \det\left[K_N(x_i, x_j)\right]_{1 \leq i,j \leq k}.$$

- Orthogonal polynomials method $\Rightarrow$ explicit expressions for $K_N(x_i, x_j)$. Let $p_N(x) = c_N x^N + \ldots$ the *Nth* orthogonal polynomial associated to $e^{-V(x)}$, the two-point kernel is

$$\begin{aligned} K_N(x,y) &= \mathrm{e}^{-\frac{(V(x)+V(y))}{2}} \sum_{k=0}^{N-1} p_k(x)\, p_k(y) \\ &= \frac{c_{N-1}}{c_N} \frac{p_N(x) p_{N-1}(y) - p_{N-1}(x)\, p_N(y)}{x-y} \mathrm{e}^{-\frac{(V(x)+V(y))}{2}}. \end{aligned}$$

- We consider Chern-Simons theory on a three-manifold M and for a gauge group G, with action

$$S(A) = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

where A is a connection on M.

- Witten showed in 1989, that the partition function of Chern-Simons theory

$$Z_k(M) = \int \mathcal{D}A e^{iS_{\mathrm{CS}}(A)},$$

defines a topological invariant.

- Chern-Simons theory is of interest in the study of (quantum) topological invariants, topological strings, ...
- The partition function of CS theory on certain manifolds has simple expressions (M. Mariño, Comm. Math. Phys. 253, 25 (2004)). The simplest case is S^3 and gauge group $U(N)$

$$Z\left(S^3\right) = \int \prod_{i<j} \left(2\sinh\frac{u_i - u_j}{2}\right)^2 \prod_{i=1}^{N} \mathrm{e}^{-u_i^2/2g_s} \frac{du_i}{2\pi}.$$

- Thus, we have N-dimensional integral expressions for Chern-Simons partition functions whose expression resemble that of random matrix theory. Using localization (after the work of Pestun), this type of result has been extended, in the last decade, to many supersymmetric gauge theories on curved manifolds.

- Ingredients: 1) Change of variables $e^{u_i} = x_i$ 2) Symmetry of the log-normal $\omega(qx) = \sqrt{q}x\omega(x)$ (when $\omega(x) = e^{-\log^2 x_i/2g_s}$), then

$$Z\left(S^3\right) = \int \prod_{i<j}\left(2\sinh\frac{u_i-u_j}{2}\right)^2 \prod_{i=1}^{N} e^{-u_i^2/2g_s}\frac{du_i}{2\pi}$$
$$= (2\pi)^{-N} e^{\frac{-N^3 g_s}{2}} \int \prod_{i=1}^{N} dx_i e^{-\frac{\log^2(x_i)}{2g_s}} \prod_{i<j}(x_i - x_j)^2 .$$

- Last expression is the Stieltjes-Wigert matrix model. For the partition function computation, we actually only need the leading coefficients $p_i(x) = a_i x^i + \dots$, which are

$$a_j = q^{(j+1/2)^2}\left\{(1-q)\dots\left(1-q^j\right)\right\}^{-1/2} .$$

- The partition function in terms of the orthogonal polynomials is:

$$Z=\frac{N!}{\prod_{i=0}^{N-1}a_i^2}=N!a_0^{-2N}\prod_{i=1}^{N-1}\left(\left(\frac{a_{i-1}}{a_i}\right)^2\right)^{N-i}.$$

- Using the coefficients, we have $\left(\frac{a_{j-1}}{a_j}\right)^2=q^{-4j}\left(1-q^j\right)$ and $a_0=q^{1/4}$, leads to

$$Z_{sw}=N!q^{-\frac{1}{6}N(2N-1)(2N+1)}\prod_{j=1}^{N-1}\left(1-q^j\right)^{N-j},$$

transforming the product term and identifying $g_s=\frac{2\pi i}{k+N}$, we have

$$Z\left(S^3\right)=\mathrm{e}^{\frac{1}{4}i\pi N^2}\left(k+N\right)^{-N/2}\prod_{j=1}^{N-1}\left(2\sin\frac{\pi j}{k+N}\right)^{N-j}.$$

Example II: Chern-Simons theory with susy matter

J. Russo, G. Silva and MT, CMP 338 (2015) 1411-1442, G. Giasemidis and MT JHEP 01 (2016) 68; MT JHEP 04 (2016) 168

- Supersymmetric Chern-Simons theory with matter also admits a matrix model rep. The case of $\mathcal{N}=2$ $U(N)$ CS with N_f massive hypermultiplets is

$$Z_{N_f}^{U(N)} = \int d^N\mu \frac{\prod_{i<j} 4\sinh^2\left(\frac{1}{2}(\mu_i-\mu_j)\right)\ e^{-\frac{1}{2g}\sum_i \mu_i^2 + i\eta\sum_i \mu_i}}{\prod_i \left(2\cosh\left(\frac{1}{2}(\mu_i+m)\right)\right)^{N_f}},$$

- With the change of variables $z_i = c\ e^{\mu_i}$, $c = e^{g\left(N-\frac{N_f}{2}\right)}$, we may write the matrix model in the form

$$Z_{N_f}^{U(N)} = C_N \int_{[0,\infty)^N} d^N z \prod_{i<j}(z_i-z_j)^2 \frac{e^{-\frac{1}{2g}\sum_i (\ln z_i)^2 + i\eta\sum_i \ln z_i}}{\prod_i \left(1+\frac{z_i e^m}{c}\right)^{N_f}}.$$

- The matrix model can now be analyzed with either orthogonal polynomials or by rewriting it as a Hankel determinant and using special functions (Mordell integrals).
- Analytical solutions can be obtained. Example:

$$Z^{U(1)}_{k,N_f=2} = -e^{-\mathrm{i}\pi(\mathrm{i}\eta+\frac{k}{4})}\, e^{-m(\mathrm{i}\eta+\frac{k}{2})+\frac{ikm^2}{4\pi}} \left(\left(-\mathrm{i}\eta - \frac{\mathrm{i}km}{2\pi} + \frac{k}{2}\right) G_+ - G'_+\right)$$

where

$$G_+ = \frac{1}{e^{2\pi\eta-km}-1}\left(-\sqrt{\frac{\mathrm{i}}{k}}\,\sum_{r=1}^{k} e^{\frac{\mathrm{i}\pi}{k}(r-\mathrm{i}\eta-\frac{k}{2}+\mathrm{i}\frac{km}{2\pi})^2} + \mathrm{i}\right).$$

- With the matrix model one also finds out a third order phase transition, of the model with N_f fundamental and $\overline{N_f}$ antifundamentals. Scaling limits: $t \equiv gN$ and $\zeta = N_f/N$ both fixed for $N \to \infty$ (+ decompactification limit). The free energy has then 3 regions $\lambda < 1$, $1 < \lambda < \zeta/(1-\zeta)$ and $\lambda > \zeta/(1-\zeta)$.
- Another feature of the gauge theory captured with the matrix model is a Seiberg-like duality in 3d, known as Giveon-Kutasov duality. In particular, the partition function satisfies

$$Z_{N_f,k}^{U(N_c)}(\eta) = e^{\mathrm{sgn}(k)\pi\mathrm{i}\left(\phi_{|k|,N_f} - \eta^2\right)} Z_{N_f,-k}^{U(|k|+N_f-N_c)}(-\eta),$$

this duality is tested/confirmed and the quadratic -in k- phase factor $\phi_{|k|,N_f}$ obtained with the matrix model.

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- Examples of random matrix models with a $V(x) = \log^2 x$ potential -and certain deformations thereof- can be solved exactly, giving CS (or CS+matter) observables. Schur polynomials and combinatorics also needed in general.
- Matrix model descriptions appear in different settings, allowing to establish relationships between seemingly unrelated objects. Phase transitions of the gauge theories can also be analyzed with matrix models.