

# The Heisenberg XX spin chain and low-energy QCD

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By using random matrix models we uncover a connection between the low energy sector of four dimensional QCD at finite volume and the Heisenberg XX model in a 1d spin chain. This connection allows to relate crucial properties of QCD with physically meaningful properties of the spin chain, establishing a dictionary between both worlds. We predict for the spin chain a third-order phase transition and a Tracy-Widom law in the transition region. We finally comment on possible numerical implications of the connection as well as on possible experimental implementations.

The strong interaction is the fundamental force of nature which describes the interaction between quarks and gluons, the elementary constituents of hadronic matter. It is described by quantum chromodynamics (QCD), a  $SU(3)$  Yang-Mills theory with a number of distinctive properties such as asymptotic freedom [1], which correctly describes that the interaction between particles becomes asymptotically weaker as distance decreases and energy increases. This crucial property, in agreement with Bjorken scaling and experimental data, is due to the negativity of the  $\beta$  function describing the variation of the coupling constant of the theory under the renormalization group flow [1]. Phenomenology also tells us that the up and down quarks are very light. The case of massless quarks implies some additional symmetries, named chiral symmetries, which would allow separate transformations between the left-handed quarks and the right-handed ones. Such behavior is not observed and hence, in a realistic QCD, the chiral symmetry must be spontaneously broken. Low energy QCD, which is the regime we are interested in, is deeply related to the notion of chiral symmetry breaking and it can be explored with chiral perturbation theory [2–4]. Recall that quarks interact weakly at high energies and strongly at low energies and, therefore, the low-energy regime is described by non-perturbative physics. Finally, chief among the features of QCD is the confinement of quarks into hadrons, either mesons ( $q\bar{q}$ ) or baryons ( $qqq$ ). Confinement in a gauge theory is usually probed by studying the behavior of Wilson loops observables [5].

There has been a lot of theoretical and numerical approaches to analyze QCD and related gauge theories, such as effective field theory and chiral perturbation theory [2, 3], lattice gauge theory [6, 7], the light-cone quantization [8], gauge-string duality and AdS/CFT approaches [9], etc. Very recently, and motivated by an idea of Feynman [10], a new route has appeared to understand Abelian and non-Abelian gauge theory: simulating it in a different controllable quantum system, such as cold atoms in optical lattices [11, 12]. Some first steps for different quantum field theories (QFT) have been carried out in [13–20].

In this paper, we initiate a different, but somehow re-

Low energy QCD	Thermal correlations (XX model)
number of flavors	number of particles
topological sector	ket vs. bra shift
$\theta$ angle	projection onto momentum $\theta$
different matter content	different boundary conditions
on the lattice	longer range interactions

TABLE I: *Dictionary* relating properties of QCD with properties of the thermal averages in the XX spin chain.

lated approach. By combining a result of Leutwyler and Smilga [21] (based on the previous seminal work on chiral perturbation theory [3, 4]) with a result of Bogoliubov et al. [22–25] we uncover a mapping between the low energy sector of QCD with thermal correlation functions in the 1D Heisenberg XX model or, via a Jordan-Wigner transformation [26], thermal correlation functions in a 1D free fermion system. The connection is made by relating both objects to a random matrix model [27]. Building upon this starting point, we are able then to relate crucial properties of QCD with physically meaningful properties of the spin chain. For instance we show that the number of flavors in QCD corresponds to the number of particles (spins down) in the 1d chain; (2) the topological charge in QCD is associated with the signature that topological 2D systems leave on their boundary theory [28–31]; (3) different matter content, such as Majorana fermions, corresponds to different boundary conditions in the spin chain; or that (4) putting QCD on the lattice enforces the addition of next-to-nearest neighbor terms in the spin chain Hamiltonian. The connection allows us to uncover also a third order phase transition in the XX model since, again via random matrix models, one can relate both low-energy QCD and the thermal correlation functions of the XX chain with the so called Gross-Witten model, a 2d Yang-Mills theory with gauge group  $U(N)$  and no matter fields which has a third order phase transition in the limit  $N \rightarrow \infty$  [32]. Finally the connection opens the possibility of using numerical methods coming from spin chains [33] to give good estimates for the partition function of low-energy QCD. It also opens a

way to measure this partition function or to observe the Gross-Witten phase transition experimentally.

### Low energy QCD as a random matrix model

Let us start by describing the derivation in [21], which applies the ideas of effective field theory [2] to the study of the meson sector of QCD [4, 21]. Recall that the main idea of an effective field theory approach is to integrate out the heavy degrees of freedom (the most massive fields) of the theory. This is implemented to study the low-energy (meson) sector of QCD, through chiral perturbation theory ( $\chi$ PT) [3] with the quark and gluon fields of QCD replaced by a set of pion fields  $U(x)$ , which describe the degrees of freedom of the pseudo Nambu-Goldstone bosons. The effective Lagrangian depends only on the pion fields and its derivatives

$$\mathcal{L} \rightarrow \mathcal{L}_{\text{eff}}(U, \partial U, \dots) = \mathcal{L}_{\text{eff}}^{(0)} + \mathcal{L}_{\text{eff}}^{(2)} + \mathcal{L}_{\text{eff}}^{(4)} + \dots$$

Chiral symmetry provides a tight constraint to the form of these terms and, in particular, the first term  $\mathcal{L}_{\text{eff}}^{(0)}$  is just a constant which is the vacuum energy of QCD in the chiral limit. The first non-trivial term is  $\mathcal{L}_{\text{eff}}^{(2)}$  and is given by [2, 3]

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{4}F^2 \text{tr} \{ \partial_\mu U^\dagger \partial^\mu U \} + \frac{1}{2}F^2 \Sigma \text{tr} \{ M (U + U^\dagger) \}, \quad (1)$$

where  $M$  is the matrix that contains the masses of the quarks (quark mass matrix) and that will be taken below to be a multiple of the identity matrix,  $F$  is the pion decay constant, and  $\Sigma$  is the chiral condensate (which describes the spontaneous chiral symmetry breaking).

Chiral perturbation theory gives an expansion in powers of  $M$ , temperature  $T$  and inverse length  $1/L$  at fixed  $\Lambda_{QCD}$  and at fixed ratios,  $M/T^2$  and  $LT$ . The scales  $T$  and  $1/L$  are treated as small quantities of order  $p$ , where  $p$  is the momentum of the pions, whereas the quark mass matrix counts as a quantity of order  $p^2$  [4]. The most general gauge invariant expression consistent with the symmetries of QCD that can be formed within the effective theory at  $\mathcal{O}(p^2)$  is given by (1). This Lagrangian holds under the condition for the volume  $V^{1/4} \gg 1/\Lambda_{QCD}$  where  $\Lambda_{QCD}$  is the length scale of QCD [4]. In this way only the Goldstone modes contribute to the mass-dependence of the partition function [4].

In addition, for quark masses for which the Compton wavelength of the Goldstone modes is much larger than the size of the box  $1/m_\pi \gg V^{1/4}$ . [84]. This is known as the epsilon regime of QCD since it is an expansion in terms of  $\varepsilon^2 \sim \frac{m_\pi}{\Lambda_{QCD}}$  [4]. The fluctuations of the zero

momentum modes of the pion fields dominate the fluctuations of the nonzero momentum modes and only the former are taken into account in the thermal average [21]. These two conditions on the volume are also referred to as the kinetic domain [34] since the kinetic term of the chiral Lagrangian can be ignored. The low-energy partition function is then [4]

$$Z_{\text{eff}}(M, \theta) = \int_{U \in SU(N_f)} dU \exp \left( V \Sigma \text{Re tr} \{ M U^\dagger \} e^{i\theta/N_f} \right),$$

since only the constant fields contribute to its mass dependence.

Note that the inverse temperature  $\beta$  of the gauge theory does not appear since, in the low-energy effective field theory, one can absorb it in the low-energy constants [4]. The appearance of the  $\theta$  parameter is because, due to the explicit breaking of the axial symmetry  $U_A(1)$ , one is naturally led to also consider the addition of a theta term to the original QCD Lagrangian [21]

$$\mathcal{L}_\theta = -\frac{i\theta}{32\pi^2} F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a,$$

where the field strength and its dual are given by [35]

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc} A_\mu^b A_\nu^c \quad \tilde{F}_{\mu\nu}^a = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta},$$

with  $f_{abc}$  the structure constants of the gauge group  $SU(N_c)$ . This is the same topological term that appears in topological insulators [36]. The topological charge

$$\nu = \frac{1}{32\pi^2} \int F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a d^4x$$

characterized by the integer  $\nu$ , is a topological invariant. The low-energy QCD partition function is then written as [21]

$$Z_{QCD}^{\text{eff}}(M, \theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z_\nu^{\text{eff}}(M). \quad (2)$$

The effective partition function at fixed  $\nu$  follows then from Fourier inversion [21]

$$Z_{\nu, N_f}^{\text{eff}}(M) = \int_{U(N_f)} dU (\det(U))^\nu \exp \left( \frac{V\Sigma}{2} \text{Tr} [(M(U + U^\dagger))] \right). \quad (3)$$

Now, using Weyl's integration formula, and assuming that  $M$  is a multiple of the identity, that is, the masses of all quarks are taken identically  $m$ , one obtains [21]

$$Z_{\nu, N_f}^{\text{eff}}(m) = \frac{1}{(2\pi)^{N_f} N_f!} \int_{-\pi}^{\pi} d\varphi_1 \cdots \int_{-\pi}^{\pi} d\varphi_{N_f} \prod_{1 \leq j < k \leq N_f} |e^{i\varphi_k} - e^{i\varphi_j}|^2 \left( \prod_{j=1}^{N_f} e^{V\Sigma m \cos(\varphi_j)} \right) \left( \prod_{j=1}^{N_f} e^{i\nu\varphi_j} \right). \quad (4)$$

Notice that this matrix model (an integral of this type with a Vandermonde term  $|e^{i\varphi_k} - e^{i\varphi_j}|^2 = 4 \sin^2(\frac{\varphi_k - \varphi_j}{2})$  is a unitary random matrix ensemble [27]), in the case  $\nu = 0$ , is the Gross-Witten matrix model, that appeared in the study of lattice 2d Yang-Mills theory with the Wilson lattice action [32][85]. In that theory there is no matter, so it corresponds to two dimensional gluodynamics. Notice however a crucial difference between the appearance of the matrix model in the two theories: in the 2d Yang-Mills theory  $N_f = 0$  and the group integration is then over  $U(N)$  which corresponds to  $U(N_c)$  in the 4d QCD case. Note that, as pointed out in [34], the matrix integration (3) is over  $U(N_f)$ , the flavor space and hence, while the model is identical, the description is very different. Taking this into account, we will also focus on the very distinctive property of the Gross-Witten matrix model [32]: a third-order phase transition in the  $N \rightarrow \infty$  limit which has been the object of intense interest over three decades, since it plays a rather paradigmatic role in the study of confinement/deconfinement and Hagedorn phase transitions [37] and has also been a guide in the study of phase transitions in 4d Yang-Mills theory [38]. We shall thus discuss both aspects of the correspondence with the spin chain: the description of low-energy QCD in terms of the spin chain and the implications of the Gross-Witten phase transition on the spin chain model.

#### *Complexity of the Leutwyler-Smilga integral*

Before proceeding to establishing and exploiting a spin chain representation of (2) and (4), we discuss some aspects of their numerical evaluation. In particular, by calling  $\beta = V\Sigma m$ , (4) is an integral representation of  $\det(I_{i-j+\nu}(\beta))_{i,j}^n$  where  $I_\nu(\beta)$  denotes the modified Bessel function of second order, which is the  $\nu$ -th Fourier coefficient of the weight function of the matrix model, namely  $e^{\beta \cos \theta}$ .

The numerical evaluation of the Bessel function for a fixed small value of the order  $\nu$  is immediate since a numerical evaluation of its integral representation with the trapezoidal rule is exponentially convergent [39].

However, its evaluation for large values of the order  $\nu$  and the argument  $\beta$  is a notoriously complex problem and the development of numerical implementations of uniform asymptotic expansions of the Bessel function in that regime is a subject of much current interest (see [43] and references therein). Indeed, even though the problem of

its evaluation goes back to Debye [40], who devised non-uniform asymptotic expansions, and also that non-trivial uniform asymptotic expansions were found in the 1950s [41, 42], it turns out that the coefficients of the asymptotic expansion not only exhibit resurgence [86] but also involve the evaluation of higher transcendental functions, in this case Airy functions.

More specifically, the direct application of the uniform asymptotics becomes problematic when the argument and the order of a Bessel function are almost equal, due to huge numerical cancellations involved in evaluating the individual coefficients in the uniform asymptotic expansions [43] (which is a consequence of the confluence of two saddles in the steepest-descent study of the integral representation of the function).

The phenomena of the appearance and coalescence of saddles is of course specially relevant when  $\beta$  is complex, due to the different ensuing crossings of Stokes lines in the steepest-descent study of the integral representation of the Bessel function [44].

At any rate, the numerical evaluation of (3) for a non-trivial topological sector (i.e. very large  $\nu$ ) is delicate at best because, in addition, standard numerical implementations of the Bessel function can underflow for large  $\nu$  [45], in which case also the posterior evaluation of the determinant with Gaussian elimination might be problematic.

This of course has the same implications for (2), where summing over all topological sectors is involved. Notice also that it is possible to further characterize (2) analytically by plugging the integral representation (4) in (2), as was done in [46]. However, the resulting expression, as expected, loses its random matrix/determinantal form and it involves the evaluation of Bessel functions *and* a posterior multivariable integration with the same weight as in (4), but with the Bessel functions in the integrand [46]. Only in the two simplest cases, corresponding to one and two flavours (one and two spins flipped, in our forthcoming picture) has the partition function an explicit analytic expression [46].

The connection made in this paper (equation (13) below) opens the possibility of using numerical methods developed in the study of spin chains, such as White's Density Matrix Renormalization Group (DMRG) algorithm [47], or more concretely some of its finite-temperature versions like [33], as an alternative method to compute the Leutwyler-Smilga integral for real  $\beta$ . For imaginary  $\beta$ , where classical simulation methods usually break down

for large  $\beta$ , one may use quantum simulations with optical lattices. Indeed, as we comment below when discussing experimental implementations, the experiment [48] does exactly the job.

### 1D XX model and thermal correlation functions

Let us now describe the result in [22–25] which relates some thermal correlation function of the XX model to a matrix model, which turns out to be the same as before.

Let us begin our discussion by presenting the spin chain model. The  $S = 1/2$  Heisenberg XX spin chain is one of the simplest integrable magnetic chains. It has a well-known mapping, using the Jordan-Wigner transformation, to a free fermion system [26]. This infinite chain (which we consider with periodic boundary conditions) is characterized by the Hamiltonian

$$\hat{H} = -\frac{1}{2} \sum_i \sigma_i^- \otimes \sigma_{i+1}^+ + \sigma_i^- \otimes \sigma_{i-1}^+ + \frac{h}{2} \sum_i (\sigma_i^z - \mathbb{1}), \quad (5)$$

where the summation is over all lattice sites and  $h > 0$ . As usual,  $\sigma_i^\pm = (\sigma_i^x \pm i\sigma_i^y)/2$ , where  $\sigma_i^x$  and  $\sigma_i^y$  together with  $\sigma_i^z$  denote the Pauli spin operators and  $h$  represents the strength of an external magnetic field. The commutation relations are

$$[\sigma_i^+, \sigma_k^-] = \sigma_i^z \delta_{ik}, \quad [\sigma_i^z, \sigma_k^\pm] = \pm 2\sigma_i^\pm \delta_{ik}.$$

These operators are nilpotent  $(\sigma_i^\pm)^2 = 0$ , a property that will lead to a determinantal form for the correlation functions that we shall focus on. The other operator satisfies  $(\sigma_i^z)^2 = 1$ .

Let us begin by defining and describing the correlation functions of the model. Thermal correlation functions of spin chains have been studied for some time [49] and are known to admit determinantal expressions which are simpler in the case of the XX model (5) and have been studied explicitly more recently [22–25]. Following [25], the correlation function will be defined on a ferromagnetic state, which is characterized by having all the spins up  $|\uparrow\rangle = \otimes_i |\uparrow\rangle_i$ , which satisfies  $\sigma_k^+ |\uparrow\rangle = 0$  for all  $k$ , and the state is also normalized  $\langle \uparrow | \uparrow \rangle = 1$ . This state is annihilated by the Hamiltonian  $\hat{H} |\uparrow\rangle = 0$  and the thermal correlation functions are defined by

$$F_{j_1, \dots, j_K; l_1, \dots, l_K}(\beta) = \langle \uparrow | \sigma_{j_1}^+ \cdots \sigma_{j_K}^+ e^{-\beta \hat{H}} \sigma_{l_1}^- \cdots \sigma_{l_K}^- | \uparrow \rangle.$$

Let us consider first the particular case where we only

have one spin down  $K = 1$ .

$$F_{jl}(\beta) = \langle \uparrow | \sigma_j^+ e^{-\beta \hat{H}} \sigma_l^- | \uparrow \rangle. \quad (6)$$

By taking into account the commutation relation

$$\begin{aligned} [\sigma_j^+, \hat{H}] &= -\frac{1}{2} \sum_k \Lambda_{jk} \sigma_j^z \sigma_k^+ - h \sum_k \sigma_k^+ \delta_{j,k} \\ &= -\frac{1}{2} \sigma_j^z (\sigma_{j-1}^+ + \sigma_{j+1}^+) - h \sigma_j^+, \end{aligned} \quad (7)$$

where  $\Lambda_{jk} = \delta_{j,k+1} + \delta_{j,k-1}$ , together with the property  $\langle \uparrow | \sigma_j^z = \langle \uparrow |$ , it follows immediately [25]

$$\begin{aligned} \frac{d}{d\beta} F_{jl}(\beta) &= -\langle \uparrow | \sigma_j^+ \hat{H} e^{-\beta \hat{H}} \sigma_l^- | \uparrow \rangle \\ &= \frac{1}{2} \langle \uparrow | (\sigma_{j-1}^+ + \sigma_{j+1}^+ + 2h\sigma_j^+) e^{-\beta \hat{H}} \sigma_l^- | \uparrow \rangle. \end{aligned}$$

Hence

$$\frac{d}{d\beta} F_{jl}(\beta) = \frac{1}{2} (F_{j+1,l}(\beta) + F_{j-1,l}(\beta)) + h F_{j,l}(\beta). \quad (8)$$

This equation is that of a symmetric random walk on a line. Let us also remark that by commuting  $\hat{H}$  with  $\sigma_l^-$ , there is an analogous difference equation but for subscript  $l$  with fixed subscript  $j$  [25]. Both equations are subject to the initial condition  $F_{jl}(0) = \delta_{jl}$ , and to boundary conditions that depend on the type of lattice considered. The results in [25] show that the case of general  $K > 1$  generalizes in a straightforward way and the multi-dimensional analogue of (8) is obtained. The initial condition is the same  $F_{j_1, \dots, j_K; l_1, \dots, l_K}(0) = \delta_{j_1 l_1} \cdots \delta_{j_K l_K}$  and the correlation function also satisfies the conditions  $F_{j_1, \dots, j_K; l_1, \dots, l_K}(\beta) = 0$  if  $l_r = l_s$  or  $j_r = j_s$  ( $r, s = 1, \dots, K$ ), due to the nilpotency of the spin operators,  $(\sigma_i^\pm)^2 = 0$ . This "non-intersecting" property suggests a determinantal structure and indeed, the solution of the equation for general  $K$  can be expressed as [25]

$$F_{j_1, \dots, j_K; l_1, \dots, l_K}(\beta) = \det_{1 \leq r, s \leq K} \{F_{j_r l_s}(\beta)\}, \quad (9)$$

where  $F_{jl}(\beta)$  are the one-particle correlation functions satisfying (8). A matrix model expression for this determinant is given by [25, 50, 51]

$$F_{j_1, \dots, j_K; l_1, \dots, l_K}(f, \beta) = \frac{1}{(2\pi)^K K!} \int_{-\pi}^{\pi} d\varphi_1 \cdots \int_{-\pi}^{\pi} d\varphi_K \prod_{1 \leq j < k \leq K} |e^{i\varphi_k} - e^{i\varphi_j}|^2 \left( \prod_{j=1}^K f(\varphi_j) \right) \overline{\hat{s}_\alpha(e^{i\varphi_1}, \dots, e^{i\varphi_K})} \hat{s}_\gamma(e^{i\varphi_1}, \dots, e^{i\varphi_K}), \quad (10)$$

where  $\hat{s}_\lambda(e^{i\varphi_1}, \dots, e^{i\varphi_n})$  is a Schur polynomial, a symmetric polynomial [52]. The relationship between the partitions  $\alpha$  and  $\gamma$  in the r.h.s. of (10) and the  $j$  and  $l$  that appear in the thermal correlation function is [25]

$$\begin{aligned} \alpha_r &= j_r - K + r \\ \gamma_r &= l_r - K + r \end{aligned}$$

and the weight function  $f(\varphi)$  in the matrix model (10) is the generating function of the one-spin flip process (8). Therefore, noticing that (10) is of the same form as (4) but more general due to the presence of two Schur polynomials, we can identify the number of flipped spins  $K$  with the number of flavors  $N_f$ . In addition,  $f(\varphi)$  being the generating function of the one-spin flip process (8), is generically given by

$$f(\beta, e^{i\varphi}) = \sum_{j=-\infty}^{\infty} F_{jl}(\beta) e^{i\varphi j},$$

and, in this particular case, by

$$f(\beta, e^{i\varphi}) = f(0, \lambda) \exp\left(\frac{\beta}{2}(2h + (e^{i\varphi} + e^{-i\varphi}))\right). \quad (11)$$

Therefore, the weight function in (10) is

$$f(\varphi) = e^{\beta(h + \cos \varphi)}. \quad (12)$$

Notice that the analysis and the final result for the case of one flipped spin  $K = 1$  is identical to Glauber's seminal study of the kinetic Ising model [53]. In particular, the thermal average (6) behaves like the expectation of a single spin in an infinite ring in [53], with the time variable in [53] identified with our  $\beta$ .

By considering now the specific pattern of flipped spins  $j_r = \nu + K - r$  and  $l_r = K - r$ , we get in (10) the partitions  $\alpha = (0, \dots, 0)$  and  $\gamma = (\nu, \dots, \nu)$ . Using that in this case [52]

$$\begin{aligned} \hat{s}_\alpha(e^{i\varphi_1}, \dots, e^{i\varphi_K}) &= 1 \\ \hat{s}_\gamma(e^{i\varphi_1}, \dots, e^{i\varphi_K}) &= \prod_{j=1}^K e^{i\nu\varphi_j}, \end{aligned}$$

we recover exactly equation (4) from equation (10). That is, we obtain the key equation

$$\langle \dots, \underbrace{\uparrow, \downarrow, \dots, \downarrow}_{N_f}, \underbrace{\uparrow, \dots, \uparrow}_{\nu}, \dots \rangle e^{-\beta \hat{H}_{\text{XX}}} \langle \underbrace{\downarrow, \downarrow, \dots, \downarrow}_{N_f}, \uparrow, \dots \rangle = Z_{\nu, N_f}^{\text{eff}}(m) \quad (13)$$

where  $\beta = V\Sigma m$ .

## A dictionary QCD – spin chains

### The topological sector

The first obvious connection arising from equation (13) is that the number of flavors  $N_f$  corresponds to the number of spins down or, equivalently in the free fermion picture, the particle-number sector to which we restrict our attention in the 1D spin chain.

On top of that, the shift  $\nu$  in the positions of the spin down particles at both sides of the thermal average in equation (13) induces a phase change in equation (4), responsible for the non-trivial topological sector of the QCD partition function. How to understand this as some type of topological order present in the XX spin chain? The question is tricky since there is in principle no clear way to define topological order in 1D. A possible answer comes from the holographic principle, where one sees a (not necessarily normalized) 1D thermal state  $e^{-\beta H}$  as the *boundary* of a 2D system. If the 2D system is topologically ordered, this should leave some signature in the 1D state. Starting with the seminal work of Li and Haldane [28], there has been several recent discussions about which this signature is [29–31, 54]. Two key facts can be extracted from there: (i) each topological sector in the bulk corresponds to projecting the thermal state of the boundary Hamiltonian in a different sector; and (ii) the bulk topology translates to some *dynamical property* on the boundary, and is hence related to the momentum. This agrees with the appearance of the translation operation  $T$  in equation (13), which can be simply restated as:

$$\begin{aligned} Z_{\nu, N_f}^{\text{eff}}(m) &= \langle \dots, \uparrow, \underbrace{\downarrow, \dots, \downarrow}_{N_f} | e^{-\beta \hat{H}_{\text{XX}}} T^{-\nu} | \underbrace{\downarrow, \dots, \downarrow}_{N_f}, \uparrow, \dots \rangle \\ &= \langle \dots, \uparrow, \underbrace{\downarrow, \dots, \downarrow}_{N_f} | e^{-\frac{\beta}{2} \hat{H}_{\text{XX}}} T^{-\nu} e^{-\frac{\beta}{2} \hat{H}_{\text{XX}}} | \underbrace{\downarrow, \dots, \downarrow}_{N_f}, \uparrow, \dots \rangle. \end{aligned}$$

The last equation resembles very much the momentum polarization tool introduced very recently in [54] as a way to detect non-trivial topological behavior [87].

To get (i) and (ii) and then show in a clearer way the topological content of  $\nu$  it is better to go back to its Fourier dual parameter  $\theta$ . In order to avoid unnecessary mathematical complications we will assume now a finite chain of  $2L + 1$  spins and define  $\hat{T} = \frac{1}{2L+1} \sum_{\nu=-L}^L e^{i\nu\theta} T^{-\nu}$ . By taking a basis of states  $|k\rangle$

with definite momentum  $T|k\rangle = e^{i\frac{2\pi k}{2L+1}}|k\rangle$  and changing variables  $\theta = \frac{2\pi\theta'}{2L+1}$  one can see that

$$\hat{T}|k\rangle = \frac{1}{2L+1} \sum_{\nu=-L}^L e^{\frac{2\pi\nu i}{2L+1}(\theta'-k)}|k\rangle = \delta_{k,\theta'}|k\rangle$$

and hence  $\hat{T}$  is just the projector  $P_\theta$  onto the states with momentum  $\theta$ . Since it commutes trivially with the Hamiltonian, we get finally

$$\begin{aligned} & \frac{1}{2L+1} \sum_{\nu=-L}^L e^{i\nu\theta} \langle \dots, \uparrow, \underbrace{\downarrow, \dots, \downarrow}_{N_f} | e^{-\beta\hat{H}_{\text{xx}}} T^{-\nu} | \underbrace{\downarrow, \dots, \downarrow}_{N_f}, \uparrow, \dots \rangle \\ &= \langle \dots, \uparrow, \underbrace{\downarrow, \dots, \downarrow}_{N_f} | P_\theta e^{-\beta\hat{H}_{\text{xx}}} P_\theta | \underbrace{\downarrow, \dots, \downarrow}_{N_f}, \uparrow, \dots \rangle. \end{aligned}$$

By considering the limit  $L \rightarrow \infty$  and (2) this has the extra benefit of giving an interpretation of the global partition function  $Z_{\text{QCD}}^{\text{eff}}(\theta)$  as a thermal average on the XX-model when the Hamiltonian  $\hat{H}$  is projected onto the sector of momentum  $\theta$ . That is,  $Z_{\text{QCD}}^{\text{eff}}(\theta) =$

$$\lim_{L \rightarrow \infty} (2L+1) \langle \dots, \uparrow, \underbrace{\downarrow, \dots, \downarrow}_{N_f} | P_\theta e^{-\beta\hat{H}_{\text{xx}}} P_\theta | \underbrace{\downarrow, \dots, \downarrow}_{N_f}, \uparrow, \dots \rangle. \quad (14)$$

A mathematically fully rigorous argument of that will be provided in the Appendix.

#### *Different matter content*

In the random matrix description of the thermal correlators one can obtain symmetries other than the unitary symmetry of (10). As happens with the analogous setting of the Calogero model [55] and of non-intersecting random walks [56, 57], the inclusion of boundaries in the problem leads to other symmetries, such as orthogonal and symplectic symmetries. One of these cases is actually treated explicitly in [25], where an absorbing boundary condition at the origin is shown to lead to the same matrix model, but with a correlation term between eigenvalues

$$\prod_{i=1}^K \sin^2 \theta_i \prod_{1 \leq j < k \leq K} \sin^2 \left( \frac{\theta_j - \theta_k}{2} \right) \sin^2 \left( \frac{\theta_j + \theta_k}{2} \right) \quad (15)$$

instead of the usual Vandermonde in (10). These other situations have a counterpart in the low-energy QCD. In the chiral limit (with the masses of the fermions  $m_f \rightarrow 0$ ) the relevant random matrix ensembles are the chiral GUE, chiral GOE and chiral GSE ensembles [88] [58], which are the ensembles that appear when the gauge

theory has  $SU(N_c)$  symmetry, with  $N_c \geq 3$ , for  $SU(2)$  gauge group, and again for  $SU(N_c)$  and  $N_c \geq 3$  but in the adjoint representation (and fermions in the adjoint representation are Majorana fermions [21]), respectively [58]. These are precisely the resulting ensembles that describe the spin chain in the limit  $\beta \rightarrow \infty$ , because the weak-coupling limit of the Gross-Witten model is a Gaussian unitary ensemble [59]. The limitation in this case is due to the fact that  $\beta$  is the parameter in the weight function of the resulting Gaussian ensemble and therefore, taking into account the identification of parameters in [58], this implies that there is a corresponding vanishing limit of the quark condensate. Thus, the correspondence in this setting is more subtle, due to the role played by Gaussian ensembles, which only emerge in our setting in the limit  $\beta \rightarrow \infty$ . Hence, these other cases have to be considered in more detail but the point is that other relevant symmetries can be in principle described by considering boundaries in the spin chain model.

#### *Effects of a lattice*

In addition to other symmetries, obtained with the inclusion of boundaries in the spin chain, one can also consider additional interactions between neighboring spins in the chain. These new interactions modify accordingly the weight function in the matrix model (10). This allows to extend the correspondence between the spin chain and low-energy QCD to the case where the gauge theory is studied on the lattice [60, 61]. The lattice breaks the chiral symmetry explicitly and hence the effects of the lattice spacing lead to new terms in chiral perturbation theory. This extended low energy theory is known as Wilson chiral perturbation theory and leads to an extension of the matrix model (3), characterized by the addition of potential terms [60, 61]

$$\begin{aligned} V(U) &= -a^2 V W_6 \text{Tr} [(U + U^\dagger)]^2 - a^2 V W_7 \text{Tr} [(U - U^\dagger)]^2 \\ &\quad - a^2 V W_8 \text{Tr} (U^2 + U^{\dagger 2}), \end{aligned} \quad (16)$$

where  $a$  denotes the lattice spacing and  $W_6, W_7$  and  $W_8$  are the new low energy constants. The first two terms in (16) are multi-trace potentials which are more difficult to treat in general and, for the moment, have no known spin chain representation. However, these terms are expected to be suppressed in the large  $N_c$  limit and are often not considered [60, 61]. Interestingly enough, the remaining potential term in (16) can be described in the same manner as above, just by generalizing the spin chain to include next-to nearest neighbors interactions. The resulting Hamiltonian is then

$$\hat{H} = -\frac{1}{2} \sum_i J_1 (\sigma_i^- \otimes \sigma_{i+1}^+ + \sigma_i^- \otimes \sigma_{i-1}^+) +$$

$$+J_2 (\sigma_i^- \otimes \sigma_{i+2}^+ + \sigma_i^- \otimes \sigma_{i-2}^+).$$

Notice that we previously have identified the  $\beta$  parameter of the spin chain with a single combination of parameters of the effective field theory:  $\beta = mV\Sigma$ . Now we have to identify  $\beta J_1 = mV\Sigma$  and  $\beta J_2 = 2a^2VW_8$ . Thus, the relative strength of the interactions at first and second neighbors depends on the quotient between the masses of the quarks and the lattice spacing, together with the respective low-energy constants

$$\frac{J_1}{J_2} = \frac{m\Sigma}{2a^2W_8}.$$

### Finite chain errors. Experimental accessibility

It is also shown in [24], with a similar argument, that in the case of a finite chain of  $L$  sites, the thermal average at the right hand side of equation (13) is nothing but the Riemann sum associated to the integral (4) when we evaluate on the vertices of a lattice division of the hypercube  $[-\pi, \pi]^{N_f}$  of length  $\frac{2\pi}{L}$ . A recent result by Baik and Liu [63] shows that the error obtained by this particular Riemann sum approximation decreases exponentially with  $L$ . More concretely, the relative error is  $O(e^{-c(L-N_f)})$  as  $L - N_f \rightarrow \infty$ , even if  $N_f$  also goes to  $\infty$ .

This opens the door to a possible experimental measure of the quantity  $Z_{\nu, N_f}^{\text{eff}}(\beta)$  as long as the experimental setup allows the following four steps: (1) implement the XX-Hamiltonian  $\sum_i \sigma_i^+ \otimes \sigma_{i+1}^- + h.c.$  (preferably with a magnetic field on the  $z$ -direction), (2) enforce the sector of  $N_f$  particles, (3) stabilize the thermal state within the sector and (4) measure the positions where the particles are. The crucial point is to realize that then the size of the chain, and the number of times that the experiment has to be done in order to approximate  $Z_{\nu, N_f}^{\text{eff}}(\beta)$  within a relative error  $\epsilon$  scales only polynomially with  $\log(\frac{1}{\epsilon})$  (we treat  $N_f$  and  $\beta$  as constants). In order to see that, we consider a magnetic field  $h \leq -2$  such that  $|\uparrow\rangle$  is the ground state. Note that by equation (12), the magnetic field only gives a factor  $e^{\beta h N_f}$  in the thermal average so we can choose its value to our convenience. By the bound on the relative error, the length of the chain needs to scale only linearly with  $\log(\frac{1}{\epsilon})$ . The first step is to restrict to the sector given by  $N_f$  spins down (particles in the free fermion picture). In this way, one gets a Hilbert space  $\mathcal{H}_{N_f}$ , whose dimension scales polynomially with  $L$ , and hence with  $\log(\frac{1}{\epsilon})$ . The next step is to stabilize the system at the desired temperature, obtaining the thermal state  $\rho_\beta$  which is nothing but  $\frac{e^{-\beta \hat{H}}}{\text{tr } e^{-\beta \hat{H}}}$  being  $\hat{H}$  the restriction of  $\hat{H}$  to  $\mathcal{H}_{N_f}$ . Since we have chosen the magnetic field for the state  $|\uparrow\rangle$  (the vacuum in the free fermion picture) to be the ground state, it is not difficult

to see that  $\text{tr } e^{-\beta \hat{H}} \leq \dim \mathcal{H}_{N_f}$ , which makes

$$\left| \langle \uparrow | \sigma_1^+ \cdots \sigma_{N_f}^+ \rho_\beta \sigma_{N_f}^- \cdots \sigma_1^- | \uparrow \rangle \right| \geq \frac{1}{\text{polylog}(\frac{1}{\epsilon})} \quad (17)$$

But the lefthand side of (17) is the probability of, given the state  $\rho_\beta$  and measuring where the three particles are, obtaining that they are in positions one to three. Since this is larger than  $\frac{1}{\text{polylog}(\frac{1}{\epsilon})}$ , the number of times one needs to make the experiment in order to get this value accurately scales also polynomially with  $\log(\frac{1}{\epsilon})$ .

It seems that ultra cold gases in optical lattices are the best system nowadays to get the required steps (1)-(4). Indeed, very recently [48], the quantity  $Z_{\nu, N_f}^{\text{eff}}(\beta)$  have been measured for chains of around 20 sites with imaginary  $\beta$ . The case of real  $\beta$  does not seem completely out of reach. Let us briefly discuss why. There are two ways of getting the XX-Hamiltonian in an optical lattice. One is to implement a 1D lattice Hard Core Boson Hamiltonian which, by considering the particle-hole degree of freedom, is exactly the XX-Hamiltonian. This (with an extra periodic confining potential) was already shown experimentally in [64], getting an array of 1D systems with a probability greater than  $\frac{1}{11M^{\frac{3}{5}}}$  of having at least one with  $M$  particles (for  $M$  small). A different route to get such Hamiltonian, proposed before in [65] and experimentally obtained in [48], is to consider atoms with a spin degree of freedom in the insulating phase. By tuning appropriate the parameters, in second-order perturbation theory one obtains the XX-Hamiltonian as the effective Hamiltonian of the system, though in a much smaller energy scale. By using the single spin addressing recently developed in [66] as done in [48] one can enforce the sector of a fixed number of particles. By the recent technique of high-resolution fluorescence imaging [67, 68], one may also measure the position of the particles. The most subtle issue is stabilizing the thermal state. Indeed, the understanding of the thermodynamical properties of ultracold gases in optical lattices is a hot topic nowadays [69], which may lead to a solution of this problem in the near future. There are at least two possible routes for that [69]. One may start with a Bose-Einstein Condensate (BEC) in thermal equilibrium which is then adiabatically loaded into the lattice potential. Even though the XX-Hamiltonian is integrable and, as shown for instance in [70, 72] thermalization without an external bath is not guaranteed, the measures made in [64] are in excellent agreement with having a thermal state [64, 71]. A second approach is to immerse the system in a reservoir with particles of a different species [69], so that we keep the number of particles constant in the lattice. Though there is no full study of the expected thermalization time, the recent estimate of [73] for the gap of the Davies Liouvillian – the one modeling the convergence to the thermal state of a system weakly coupled to a thermal bath – in the case of a fermion hopping on a line, allows one to be

optimistic in this direction.

### A third order phase transition on the XX chain and the Tracy-Widom law

The final implication of (13) is the existence of a third order phase transition hidden in the XX-model [89] – the so called Gross-Witten transition. Let us recall here that, as was shown in the seminal paper [32], if we consider the t'Hooft parameter  $\lambda = K/\beta$ , and we make  $K \rightarrow \infty$  while keeping  $\lambda$  constant, we obtain a double-scaling limit in  $Z_{\nu=0,K}(\frac{\beta}{\sqrt{\Sigma}})$  – now we call it  $Z_{\text{GW}}(\beta, U(K))$  since it is the partition function of the Gross-Witten model with gauge group  $U(K)$  – with a third-order phase transition between the two regimes. Formally, the limit for the free energy

$$F_K(\lambda) = \frac{1}{K^2} \ln Z_{\text{GW}}(\beta, U(K))$$

gives us [32]

$$\lim_{K \rightarrow \infty} F_K(\lambda) = \begin{cases} \frac{1}{4\lambda^2} & , \lambda \geq 1 \\ \frac{1}{\lambda} + \frac{1}{2} \ln \lambda - \frac{3}{4} & , \lambda < 1 \end{cases}$$

By (13), which now reads

$$\langle \dots, \underbrace{\uparrow, \downarrow, \dots, \downarrow, \downarrow}_K e^{-\beta \hat{H}_{\text{XX}}} \underbrace{\downarrow, \downarrow, \dots, \downarrow, \uparrow, \dots}_K \rangle = Z_{\text{GW}}(\beta, U(K)) \quad (18)$$

this is a phase transition in the XX model. Notice that now the correspondence is between the number of flipped spins and the rank of the gauge group. It is noteworthy that the above mentioned exponentially small error for chains of finite size  $L$  also holds in the double-scaling limit [63, 74]. In particular, for the two phases, it holds that, if  $R(L, K, \beta)$  denotes the Riemann sum, which is the partition function of the finite size spin chain, divided by the multiple integral (the partition function of the infinite chain), then [74]  $R(L, K, \beta) = 1 + O(e^{-cK})$  if  $L > (1 + \epsilon)\mu(K, \beta)$  (with  $\epsilon > 0$ ), where

$$\mu(K, \beta) := \begin{cases} 2\sqrt{K\beta}, & \lambda < 1, \\ K + \beta, & \lambda \geq 1. \end{cases} \quad (19)$$

Note the different scaling of the finite length  $L$  in terms of the number of flipped spins and the inverse temperature, depending on the phase. This property is a direct consequence of the discreteness of the associated random matrix ensemble.

On the other hand, one may argue that this phase transition only happens in a very unnatural limit of the spin chain parameters. However, there are signatures of this phase transition (a crossover) for very small values of  $K$ , as can be seen from the plots for  $K = 1, 2, 3$  in [75]. Actually, stronger results are available, since the following estimate holds in the strong-coupling phase ( $\lambda \geq 1$ ) [76]

$$|F_K(\lambda) - F(\lambda)| \leq Ce^{-cK}, \quad (20)$$

for some constants  $C$  and  $c$  and  $F(\lambda) = \lim_{K \rightarrow \infty} F_K(\lambda)$ . This predicts an exponentially small departure, in the strong-coupling phase, for the case of a finite number of flipped spins  $K$ .

Notice that this result holds for the free energy, which is the logarithm of the thermal correlator, and not just the partition function. The other phase is a bit more delicate to analyze and it is known that the finite rank case has  $1/K$  and higher-order corrections [77]. The resummation of all the infinitely many terms is of course a complex and delicate issue, but more recent results show that the departure with the infinite rank case decays quickly also in the weak-coupling phase [78].

Thus, taking into account the results on finite chain errors and the comments above on experimental accessibility one may be able to observe the Gross-Witten phase transition experimentally in a spin chain.

Besides the work of Gross and Witten, the random matrix ensemble (4) with  $\nu = 0$  is central in the ground breaking description of the asymptotics of the length of the longest increasing subsequence in random permutations [76]. In [76], it was proved that writing

$$K = \beta + x(\beta/2)^{1/3} \quad (21)$$

and for  $\beta \rightarrow \infty$ , then

$$e^{-\beta^2/4} Z_{\text{GW}}(\beta, U(K)) \rightarrow F_2(x), \quad (22)$$

where  $F_2(x)$  is the celebrated Tracy-Widom distribution [79] which can be given in terms of the Fredholm determinant of an Airy kernel or through an integral representation involving a solution of the Painlevé II equation, from which asymptotic expansions for  $F_2(x)$  follow [27, 79]. Notice that the factor  $e^{-\beta^2/4}$  in (22) is the normalization constant of the matrix model in the limit  $K \rightarrow \infty$ . In this way the l.h.s. of (22) is actually

$$F_{\{K-r\}_{r=0}^{K-1}, \{K-r\}_{r=0}^{K-1}}(\beta) \quad \text{with} \quad \lim_{K \rightarrow \infty} F(\beta, h^*) = 1.$$

If the spin chain does not have the magnetic field term in (5) then one has to add the prefactor in (22) by hand. With the magnetic field  $h$  there is an overall prefactor  $\exp(h\beta K)$  outside the matrix model integral representation and, hence, to observe the Tracy-Widom shape which emerges in the scaling region above, one can fine-tune the magnetic field to  $h^* = \frac{\beta}{4K} \sim 1/4$ .

Notice that the result above zooms in the  $\lambda = 1$  region where the 3rd order phase transition occurs, and it describes typical and small fluctuations of order  $\mathcal{O}(K^{-2/3})$  in the transition point [80][90]. The relevance of the Tracy-Widom distribution resides on its large universality and the universal fluctuations that it characterizes have been measured in recent experiments studying



the height distribution of interfaces, in particular in the slow combustion of paper and in turbulent liquid crystals [81, 82]. Note that it makes its appearance in 2d classical statistical mechanics systems [83] whereas we are proposing that a corresponding result also holds in a quantum 1d system, the Heisenberg XX model.

### Conclusions

We have uncovered a connection between QCD and the XX model using random matrix models which allows to establish a dictionary between both worlds as sketched in Table I. This opens an avenue to connect different QFT with 1D spin chain Hamiltonians. Specially interesting is the case of Chern-Simons theory due to its connections with topology, knot theory and the fractional quantum Hall effect. We will make a full study in a forthcoming paper, showing that Table I also applies, albeit with a different 1D spin Hamiltonian.

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- [84]  $m_\pi$  denotes the mass of the lightest Goldstone boson and can be expressed in terms of the low-energy constants  $F$ ,  $\Sigma$  and the masses of the quarks. See [21] or the review [34] for details.
- [85] For  $\nu \neq 0$  the matrix model describes the determinant of a Wilson loop (P. Rossi, Phys. Lett. B 117, 72 (1982)) instead of the partition function of the 2d Yang-Mills theory.
- [86] Resurgence makes reference to the fact that the coefficients  $a_n$  of a Poincaré asymptotic series have the following special feature: the coefficients in the asymptotic expansions of  $a_n$ , as  $n \rightarrow \infty$  are equal to themselves or related to the coefficients  $a_m$ .
- [87] Following [54], one can see how (13) relates directly with the generator  $L_0$  of the Virasoro algebra associated with the CFT of the XX-model. This emphasizes, in a way

- similar to [54], the topological content of  $\nu$
- [88] Gaussian orthogonal, unitary and symplectic ensembles, respectively [27].
- [89] Third order phase transitions are rare in condensed matter systems. Another third order phase transition was found recently in [62]
- [90] Notice that we can write the condition (21) as  $x = 2^{1/3} \frac{\lambda-1}{K^{2/3}}$ , where  $\lambda$  is the 't Hooft parameter.

### Appendix: Proof of (14)

We will use some simple Banach space tools and notations for that. Let us recall that  $\ell_1$  is the Banach space of sequences  $x = (x_n)_{n \in \mathbb{Z}}$  such that  $\|x\|_1 = \sum_{n=-\infty}^{\infty} |x_n| < \infty$ . It is clear that the direct sum (or cartesian product) of a finite number of copies of  $\ell_1$ ,  $\oplus_{k=1}^K \ell_1$ , can be seen as another  $\ell_1$  just considering the norm  $\|(x^1, \dots, x^K)\| = \sum_{k=1}^K \|x^k\|_1$ . Given a sequence  $(x^k)_{k \in \mathbb{N}} \subset \ell_1$  one says that it converges to  $x \in \ell_1$  if  $\|x^k - x\|_1 \rightarrow 0$ . In a direct sum as above, convergence is simply equivalent to convergence in each of the factors. We will use the following characterization of convergence for positive sequences in  $\ell_1$ : given a sequence  $(x^k)_{k \in \mathbb{N}} \subset \ell_1$  such that  $x_n^k \geq 0$  for all  $k, n$ ,  $\lim_k x_n^k = x_n$  for all  $n$ , and  $\lim_k \sum_{n=-\infty}^{\infty} x_n^k = \sum_{n=-\infty}^{\infty} x_n$ , then  $x^k$  converges in norm to  $x \in \ell_1$ .

Let us consider  $N_f \times N_f$  matrices with values in  $\ell_1$ . We identify each column of the matrix with  $X_1 = \oplus_{k=1}^{N_f} \ell_1$ , which is another  $\ell_1$ . The pointwise determinant

$$\text{Det}_1 : \underbrace{X_1 \times \dots \times X_1}_{N_f} \rightarrow \ell_1,$$

is a continuous multilinear map on  $X_1$  with values on  $\ell_1$ .

We fix  $\beta = V\Sigma m$  and consider  $\nu$  as the variable (getting sequences in  $\nu \in \mathbb{Z}$ ).

We denote

$$R_L(\nu) = \chi_{[-L, L]}(\nu) \langle \dots, \underbrace{\uparrow, \downarrow, \dots, \downarrow}_{N_f} \rangle e^{-\beta \hat{H}_{\text{XX}}} T^{-\nu} \langle \underbrace{\downarrow, \dots, \downarrow}_{N_f}, \dots, \uparrow, \dots \rangle$$

and

$$R(\nu) = \lim_{L \rightarrow \infty} R_L(\nu) (= Z_\nu^{\text{eff}}(m)) \quad (23)$$

From the results in [21, 24] (see also (9)) we know that both  $R_L$  and  $R$  are the determinant of a Toeplitz matrix

$$R_L(\nu) = \chi_{[-L,L]}(\nu) \begin{vmatrix} q_\nu^L(\beta) & q_{\nu+1}^L(\beta) & \cdots & q_{\nu+K-1}^L(\beta) \\ q_{\nu-1}^L(\beta) & q_\nu^L(\beta) & \cdots & q_{\nu+K-2}^L(\beta) \\ \cdots & \cdots & \cdots & \cdots \\ q_{\nu-K+1}^L(\beta) & q_{\nu-K+2}^L(\beta) & \cdots & q_\nu^L(\beta) \end{vmatrix}, \quad R(\nu) = \begin{vmatrix} I_\nu(\beta) & I_{\nu+1}(\beta) & \cdots & I_{\nu+K-1}(\beta) \\ I_{\nu-1}(\beta) & I_\nu(\beta) & \cdots & I_{\nu+K-2}(\beta) \\ \cdots & \cdots & \cdots & \cdots \\ I_{\nu-K+1}(\beta) & I_{\nu-K+2}(\beta) & \cdots & I_\nu(\beta) \end{vmatrix}$$

where  $I_k(\beta)$  is the Bessel function of imaginary argument and

$$q_k^L(\beta) = \frac{1}{2L+1} \sum_{s=-L}^L e^{\frac{2\pi i s k}{2L+1}} e^{\beta \cos \frac{2\pi s}{2L+1}}$$

which is trivially periodic (in  $k$ ) with period  $2L+1$ . As commented in the finite chain analysis in the main text, the first expression is nothing but a Riemann sum associated with the integral representation of the Bessel function  $I_k(\beta)$ , which shows that  $\lim_L q_k^L(\beta) = I_k(\beta)$  for all  $k \in \mathbb{Z}$ . Moreover,  $I_k(\beta) \geq 0$ ,  $q_k^L(\beta) \geq 0$ , and we have the equalities

$$\sum_{k=-L}^L q_k^L(\beta) = e^\beta = \sum_{j=-\infty}^{\infty} I_j(\beta). \quad (24)$$

By the characterization given above, for any fixed  $r \in \mathbb{Z}$ , we get that  $(\chi_{[-L,L]}(\nu) q_{r+\nu}^L(\beta))_L \subset \ell_1$  converges to  $(I_{r+\nu}(\beta))_\nu \in \ell_1$ . Now, using the continuity of  $\text{Det}_1$ , we get that  $R_L$  converges to  $R$  on  $\ell_1$ . This implies trivially weak convergence, that is, for any bounded sequence

$(y_\nu)_{\nu \in \mathbb{Z}}$ ,

$$\lim_{L \rightarrow \infty} \sum_{\nu=-\infty}^{\infty} y_\nu R_L(\nu) = \sum_{\nu=-\infty}^{\infty} y_\nu R(\nu).$$

By taking  $y_\nu = e^{i\theta\nu}$  we get

$$\begin{aligned} Z_{QCD}^{\text{eff}}(\theta) &= \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z_\nu^{\text{eff}}(m) = \lim_{L \rightarrow \infty} \sum_{\nu=-L}^L e^{i\nu\theta} R_L(\nu) \\ &= \lim_{L \rightarrow \infty} \sum_{\nu=-L}^L e^{i\nu\theta} \underbrace{\langle \dots, \uparrow, \downarrow, \dots, \downarrow \rangle}_{N_f} e^{-\beta \hat{H}_{\text{xx}}} T^{-\nu} \underbrace{|\downarrow, \dots, \downarrow, \uparrow, \dots\rangle}_{N_f} \\ &= \lim_{L \rightarrow \infty} (2L+1) \underbrace{\langle \dots, \uparrow, \downarrow, \dots, \downarrow \rangle}_{N_f} P_\theta e^{-\beta \hat{H}_{\text{xx}}} P_\theta \underbrace{|\downarrow, \dots, \downarrow, \uparrow, \dots\rangle}_{N_f}, \end{aligned}$$

which finishes the argument.