

# CHERN-SIMONS MATRIX MODELS, TWO-DIMENSIONAL YANG-MILLS THEORY AND THE SUTHERLAND MODEL

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ABSTRACT. We derive some new relationships between matrix models of Chern-Simons gauge theory and of two-dimensional Yang-Mills theory. We show that  $q$ -integration of the Stieltjes-Wigert matrix model is the discrete matrix model that describes  $q$ -deformed Yang-Mills theory on  $S^2$ . We demonstrate that the semiclassical limit of the Chern-Simons matrix model is equivalent to the Gross-Witten model in the weak coupling phase. We study the strong coupling limit of the unitary Chern-Simons matrix model and show that it too induces the Gross-Witten model, but as a first order deformation of Dyson's circular ensemble. We show that the Sutherland model is intimately related to Chern-Simons gauge theory on  $S^3$ , and hence to  $q$ -deformed Yang-Mills theory on  $S^2$ . In particular, the ground state wavefunction of the Sutherland model in its classical equilibrium configuration describes the Chern-Simons free energy. The correspondence is extended to Wilson line observables and to arbitrary simply-laced gauge groups.

## 1. INTRODUCTION AND SUMMARY OF RESULTS

Matrix models have been a subject of much interest in gauge theory for over three decades [1]. In this paper we study random matrix models [2] related to Chern-Simons theory [3], and both ordinary two-dimensional Yang-Mills theory [4] and its  $q$ -deformed counterpart [5, 6]. We will develop some new relationships between the Chern-Simons matrix models and some of the matrix models that appear in two-dimensional Yang-Mills theory.

Let us begin by giving the definition of the joint probability distribution  $P(M)$  for the matrix elements of an  $N \times N$  matrix  $M$  [2],

$$(1.1) \quad P(M) = Z_N^{-1} \exp(-\text{Tr } V(M)) ,$$

with any potential  $V(M)$  such that the partition function  $Z_N$  exists. The integration of (1.1) over parameters related to the eigenvectors of  $M$  leads to the well-known joint probability distribution of the eigenvalues [2]

$$(1.2) \quad P(x_1, \dots, x_N) = Z_N^{-1} \prod_{i < j} |x_i - x_j|^\beta \prod_{i=1}^N \exp(-V(x_i)) .$$

The level repulsion in (1.2), described by the Vandermonde determinant, originates as the Jacobian of the transformation when passing from integration over independent matrix elements to integration over the smaller space of  $N$  eigenvalues. The integer  $\beta = 1, 2, 4$  describes the symmetry of the ensemble (orthogonal, unitary and symplectic, respectively).

These symmetries have been extended using the Cartan classification of symmetric spaces [7] (see also [8, 9, 10]). For example, the possible Jacobians of the transformations to radial coordinates are given by [8]

$$(1.3) \quad \begin{aligned} J^{(0)}(x) &= \prod_{\alpha \in \Delta^+} (x_\alpha)^{m_\alpha} , \\ J^{(-)}(x) &= \prod_{\alpha \in \Delta^+} (a^{-1} \sinh(x_\alpha))^{m_\alpha} , \\ J^{(+)}(x) &= \prod_{\alpha \in \Delta^+} (a^{-1} \sin(x_\alpha))^{m_\alpha} , \end{aligned}$$

for the various types of symmetric spaces with zero, negative and positive constant curvature, respectively, and for an arbitrary non-zero constant  $a$ . The products are taken over all positive roots of the restricted root lattice, with  $m_\alpha$  the multiplicity of the root vector  $\alpha \in \Delta^+$ , and  $x_\alpha := (x, \alpha) = \sum_i x_i \alpha^i$  are canonical coordinates on a maximal abelian subalgebra of the tangent space.

The case of the  $A_{N-1}$  root system and zero curvature leads to the well-known Gaussian matrix model distribution (we write only the Hermitian case)

$$(1.4) \quad P^{(0)}(x_1, \dots, x_N) = Z_N^{-1} \prod_{i < j} (x_i - x_j)^2 \prod_{i=1}^N \exp(-x_i^2/2) ,$$

whereas the same root lattice but in the case of negative curvature yields

$$(1.5) \quad P^{(-)}(x_1, \dots, x_N) = Z_N^{-1} \prod_{i < j} \sinh^2(a(x_i - x_j)) \prod_{i=1}^N \exp(-x_i^2/2) .$$

In the B-model topological string theory, this is the matrix model that describes  $U(N)$  Chern-Simons gauge theory on  $S^3$  [11, 12] if  $a = \sqrt{g_s}$ , where  $g_s$  is the string coupling constant which is related to the usual integer Chern-Simons level  $k$  by

$$(1.6) \quad g_s = \frac{2\pi i}{k + N} .$$

The exponential mapping  $x_i = \log u_i/a$  brings (1.5) into the usual form (1.2) (with  $\beta = 2$ ), and hence one can apply the orthogonal polynomial method of random matrix theory [2] to solve the model [13, 14]. The orthogonal polynomials here are the Stieltjes-Wigert polynomials [13, 14, 15] or, if we work with a unitary matrix model [16], the Rogers-Szegő polynomials [15].

Chern-Simons theory on  $S^3$  is equivalent to  $q$ -deformed Yang-Mills theory on  $S^2$ , with the identification  $q := e^{-g_s}$ . This result can be derived at weak coupling either directly via localization of the three-dimensional gauge theory by regarding  $S^3$  as a Seifert manifold through the Hopf fibration  $S^3 \rightarrow S^2$  [17, 18], or by recasting the two-dimensional gauge theory as a sum over instantons and explicitly demonstrating its equivalence with the Chern-Simons matrix model [19, 20]. At weak coupling, the two-dimensional  $U(N)$  gauge theory thus reproduces the perturbative A-model topological string partition function  $Z_{\text{top}}$  for the resolved conifold geometry in the large  $N$  limit. The Stieltjes-Wigert polynomial is the average of the characteristic polynomial in the matrix model, and hence describes B-brane amplitudes on the conifold [21].

In the next section we show that  $q$ -integration of the Stieltjes-Wigert matrix model directly gives the discrete matrix model that describes the strong coupling expansion of  $q$ -deformed two-dimensional Yang-Mills theory. In the large  $N$  limit, the strong coupling series has zero radius of convergence [19]. This is in sharp contrast to ordinary (undeformed) Yang-Mills theory on  $S^2$ , which undergoes a third order phase transition at large  $N$  [22] and possesses a double

scaling limit which lies in the universality class of the Gross-Witten unitary matrix model [23]. The double scaling limit of the Gross-Witten model is also of interest in the study of unitary matrix models of string theory in zero dimensions [24], of the solution of  $SU(2)$  Seiberg-Witten theory [25], and of Type 0A and 0B string theories in one dimension [26] where it is argued to describe the universality class of pure two-dimensional supergravity.

In the next section we will demonstrate that both the weak and strong coupling limits of the finite  $N$  Chern-Simons matrix models are also governed by the Gross-Witten model [27]. Recall that this is the unitary one-matrix model which arises as the one-plaquette reduction of the combinatorial quantization of Yang-Mills theory in *infinite* spacetime. In two dimensions the reduction is exact and described by the partition function [27]

$$(1.7) \quad \begin{aligned} Z_N^{\text{GW}}(\alpha) &:= \int_{U(N)} dU \exp\left(-\alpha \text{Tr}(U + U^\dagger)\right) \\ &= \int_0^{2\pi} \prod_{i=1}^N d\theta_i e^{-2\alpha \cos \theta_i} \prod_{i < j} \sin^2\left(\frac{\theta_i - \theta_j}{2}\right), \end{aligned}$$

where  $dU$  denotes the bi-invariant Haar measure for integration over the unitary group  $U(N)$ . This matrix model belongs to the class of symmetric spaces associated with the  $A_{N-1}$  root system and positive curvature.

We will thus show that the  $q \rightarrow 1$  limit of the unitary Chern-Simons matrix models are equivalent to two-dimensional Yang-Mills theory on the *plane*  $\mathbb{R}^2$ . Identifying  $\alpha = \frac{1}{g_s} = \frac{k+N}{2\pi i}$  and using the exact integration of (1.7) in the weak-coupling phase [28, eq. (5.25)], we can write down a simple all-orders expression for the  $U(N)$  Chern-Simons partition function  $Z_{N,k}(S^3)$  on  $S^3$  in the weak-coupling limit as

$$(1.8) \quad \lim_{k \rightarrow \infty} Z_{N,k}(S^3) = \left(\frac{2\pi i e^{2(k+N)/\pi i}}{N(k+N)}\right)^{N^2/2} \left(1 + (-1)^{-N^2/2} e^{-N(k+N)/\pi i}\right)^N.$$

This exact expression should reproduce the topological string partition function  $Z_{\text{top}}$  in the large  $N$  limit to leading orders of perturbation theory.

As the unitary one-matrix model (1.7) undergoes a phase transition to a strong coupling phase in the large  $N$  limit, while the  $q$ -deformed gauge theory on  $S^2$  always remains in its weak coupling phase, the nature of this relationship must be drastically different at strong coupling. We will show that the  $q \rightarrow 0$  limit of the unitary Chern-Simons matrix model is equivalent to Dyson's circular ensemble, together with an infinite tower of higher Casimir deformations corresponding to multicritical extensions [24], the lowest order of which is described by the Gross-Witten model. These deformations are analogous to those which arise in Yang-Mills theory on the noncommutative torus [29, 30]. This feature puts the  $q$ -deformed gauge theory into the context of noncommutative deformations of Yang-Mills theory [31].

In the last section we show that the celebrated Sutherland model [32], a central model in the theory of one-dimensional integrable systems, is also directly related to Chern-Simons gauge theory. This provides another connection between two-dimensional Yang-Mills theory and Chern-Simons theory, and moreover between integrable models and Chern-Simons theory on the three-sphere  $S^3$ . The Sutherland model is the exactly solvable system on a circle of circumference  $L$  defined by the  $N$ -particle quantum Hamiltonian operator [32]

$$(1.9) \quad H = -\sum_{i=1}^N \frac{\partial^2}{\partial q_i^2} + \frac{2\lambda(\lambda-1)\pi^2}{L^2} \sum_{i < j} \left(\sin \frac{\pi(q_i - q_j)}{L}\right)^{-2},$$

with the ground state wavefunction

$$(1.10) \quad \Psi_0(q_1, \dots, q_N; \lambda, L) = \prod_{i < j} \left( \sin \frac{\pi(q_i - q_j)}{L} \right)^\lambda.$$

The relationship follows immediately when the model is considered in a simple fixed crystalline configuration. As we will discuss, this is not an artificial configuration, and it corresponds to the classical equilibrium state of the Sutherland model which is naturally reached in the strong coupling limit  $\lambda \rightarrow \infty$ . We will show that the probability density distribution, evaluated in an equally spaced configuration, reproduces the partition function for Chern-Simons gauge theory on  $S^3$  with gauge group  $SU(N)$ , which is one of the simplest quantum topological invariants that can be obtained from Chern-Simons theory [3]. We demonstrate that this correspondence extends to Wilson line observables, and hence to other quantum topological invariants, by considering excited states of the Sutherland model associated to particular non-equilibrium configurations of the particles. It also holds between generalized Calogero-Sutherland models and Chern-Simons theory with other simply-laced gauge groups (associated to orthogonal and symplectic ensembles).

Recall that the Chern-Simons partition function  $Z_{N,k}(S^3)$  (or equivalently the free energy  $F_{N,k}(S^3) = \log Z_{N,k}(S^3)$ ) is central in the study of topological strings, for example. The matrix model description is useful in topological string theory, and its physical applications have been exploited mainly in that context thus far. Although there have been extensive applications of Chern-Simons theory to condensed matter physics (most notably to the fractional quantum Hall effect), our application here deals with a many-body description of Chern-Simons theory in its non-abelian version, at least for certain simple three-manifolds such as  $S^3$ . Given the natural appearance of random matrix theory, and its connection with exactly solvable models in one dimension [14] and with Laughlin wavefunctions [33], it seems worthwhile to further explore this many-body description in itself and its possible role in condensed matter physics.

This correspondence also gives an intriguing new relationship between two-dimensional Yang-Mills theory and certain one-dimensional integrable systems, which is rather different in spirit from previous relationships [34, 35] that considered wavefunctions of Calogero-Sutherland and Calogero-Moser models as reductions of those for the undeformed gauge theory on a cylinder with appropriate Wilson line insertions. The  $q$ -deformed gauge theories on  $S^2$  may thus provide computationally useful means for exploring various aspects of one-dimensional exactly solvable models. In particular, as the large  $N$  deformed gauge theory on  $S^2$  exhibits no phase transitions, there is a natural and simple well-defined large  $N$  field theory limit of the Sutherland system which is equivalent to a generalized Calogero-Sutherland model based on the  $su(\infty)$  Lie algebra. In [34] this limit is instead realized by replacing  $SU(\infty)$  with the centrally extended  $SU(N)$  loop group, which is related to Yang-Mills theory on an infinite cylinder with Wilson line insertions. It would be interesting to better understand the explicit relationship between the two approaches, given the known equivalence between the partition functions of  $q$ -deformed Yang-Mills theory on  $S^2$  and of ordinary Yang-Mills theory on a cylinder with trivial holonomies around the two boundary circles of the cylinder [36].

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## 2. FROM CHERN-SIMONS THEORY TO THE GROSS-WITTEN MODEL

**2.1. Another derivation of  $q$ -deformed Yang-Mills theory.** Let us begin by computing the  $q$ -integration of the Stieltjes-Wigert matrix model. We will use Jackson's integral, which in the single variable case is given by [37, 38]

$$(2.1) \quad \int_0^\infty d_q u w(u) = (1-q) \sum_{n=-\infty}^{\infty} w(q^n) q^n .$$

If the function  $w(u)$  is the weight function of the Stieltjes-Wigert matrix model, i.e. the log-normal distribution  $w(u) = \exp(-\log^2 u/2g_s)$ , then the  $q$ -integration gives

$$(2.2) \quad \int_0^\infty d_q u e^{-\log^2 u/2g_s} = (1-q) \sum_{n=-\infty}^{\infty} q^{n^2/2+n} ,$$

with the usual identification  $q = e^{-g_s}$ . The right-hand side of (2.2) is proportional to the theta-function

$$(2.3) \quad \Theta_{00}(z|q) = \sum_{n=-\infty}^{\infty} q^{n^2/2} z^n$$

at the particular value  $z = q$ . This theta-function is also the weight function of the unitary matrix model that describes Chern-Simons gauge theory [16, 15]. We shall see below how this feature relates the Chern-Simons matrix model with the Gross-Witten model.

Let us now compute the  $q$ -integration of the log-normal weight function in the multivariable case. For the Stieltjes-Wigert matrix model, we then have

$$(2.4) \quad \int_0^\infty \prod_{i=1}^N \frac{d_q u_i}{2\pi} e^{-\log^2 u_i/2g_s} \prod_{i<j} (u_i - u_j)^2 \\ = \left( \frac{1-q}{2\pi} \right)^N \sum_{n_1, \dots, n_N = -\infty}^{\infty} q^{\sum_i (n_i^2/2+n_i)} \prod_{i<j} (q^{n_i} - q^{n_j})^2 ,$$

which coincides with the partition function of the  $q$ -deformed two-dimensional Yang-Mills model on  $S^2$  with gauge group  $U(N)$  [5, 6]. Recall that, using solutions of the log-normal moment problem, the equivalence between the continuum and discrete matrix models was demonstrated in this case in [39], and it reads

$$(2.5) \quad \left( \frac{g_s}{2\pi} \right)^{-N/2} \int_{-\infty}^{\infty} \prod_{i=1}^N \frac{dx_i}{2\pi} e^{-x_i^2/2g_s} \prod_{i<j} \left( 2 \sinh \left( \frac{x_i - x_j}{2} \right) \right)^2 \\ = A_N(q) \sum_{n_1, \dots, n_N = -\infty}^{\infty} e^{-\frac{g_s}{2} \sum_i n_i^2} \prod_{i<j} \left( 2 \sinh \left( \frac{g_s}{2} (n_i - n_j) \right) \right)^2$$

where

$$(2.6) \quad A_N(q) := \left( \frac{q^{-(1-2N+3N^2)/2}}{(-q^{3/2-N})_\infty (-q^{N-1/2})_\infty (q)_\infty} \right)^N ,$$

with  $(q)_0 := 1$ ,  $(q)_m := \prod_{1 \leq j \leq m} (1 - q^j)$  for  $m \in \mathbb{N}$ , and  $(q)_\infty := \prod_{j \in \mathbb{N}} (1 - q^j)$ . Hence  $q$ -integration and ordinary integration of the Stieltjes-Wigert distribution gives the same result (up to overall normalization), the  $U(N)$  Chern-Simons partition function on  $S^3$ .

**2.2. Weak-coupling limit.** We will now demonstrate that the Chern-Simons matrix model yields, in the weak-coupling limit  $g_s \rightarrow 0$  ( $q = e^{-g_s} \rightarrow 1$ ), the Gross-Witten model (1.7) with coupling  $g = \alpha^{-1} \rightarrow 0$ . For this, we note that the distinctive log-square behaviour of the weight function  $w(u) = \exp(-\log^2 u/2g_s)$  for the Stieltjes-Wigert matrix model also constitutes the natural definition of  $q$ -growth [40, 41]. We use results of [40] to study its relationship with the different definitions of  $q$ -exponential functions, and in turn to better understand its  $q \rightarrow 1$  limit. The crux of our analysis is that the weight function of the Stieltjes-Wigert matrix model can be given in terms of  $q$ -exponential functions, but without the usual rescaling  $u \rightarrow (1-q)u$  that is necessary to recover ordinary calculus from  $q$ -deformed calculus when  $q \rightarrow 1$  [37, 38].

As we mentioned in the previous section, the matrix models that appear in Chern-Simons theory can be solved exactly by using orthogonal polynomials. These orthogonal polynomials are completely characterized by a discrete scaling symmetry satisfied by the weight function [39]

$$(2.7) \quad w(qu) = \sqrt{q} u w(u) .$$

This self-similarity property is called the  $q$ -Pearson equation, and it uniquely determines the orthogonal polynomials up to normalization [39].

Let us begin by briefly reviewing  $q$ -deformed expressions for exponential functions and how they tend to their classical counterparts in the limit  $q \rightarrow 1$  [37, 38]. For  $u \in \mathbb{C}$ , the two main definitions of a  $q$ -exponential function in the literature are given by

$$(2.8) \quad e_q(u) = \sum_{j=0}^{\infty} \frac{u^j}{(q)_j} = \frac{1}{(u; q)_{\infty}} ,$$

$$E_q(u) = \sum_{j=0}^{\infty} \frac{q^{j(j-1)/2} u^j}{(q)_j} = (-u; q)_{\infty} = (e_q(-u))^{-1} ,$$

where  $(u; q)_{\infty} := \prod_{j \in \mathbb{N}_0} (1 - u q^j)$ . The connection with the usual exponential function is given by the functional equation

$$(2.9) \quad e_q(u) E_q(-u) = 1$$

and the weak-coupling limits

$$(2.10) \quad \lim_{q \rightarrow 1} e_q((1-q)u) = \lim_{q \rightarrow 1} E_q((1-q)u) = e^u .$$

We now recall that the functional equation (2.7) solves the matrix model [39]. Furthermore, one can show [40] that the entire function  $f(u) = E_q(u)$  satisfies  $(1+u)f(qu) = f(u)$  and the entire function  $g(u) = e_{q^{-1}}(-u^{-1})$  satisfies  $u g(u) = (1+u)g(u)$ . It follows that the entire function

$$(2.11) \quad h(u) = f(u)g(u) = (-u; q)_{\infty} (-qu^{-1}; q)_{\infty}$$

is a solution of the  $q$ -difference equation  $u h(qu) = h(u)$ . By replacing  $q \rightarrow q^{-1}$  and rescaling  $u \rightarrow qu$ , the functional equation for  $h(u)$  reads

$$(2.12) \quad h(qu) = qu h(u) .$$

This is essentially the  $q$ -Pearson equation (2.7), but with  $q$  instead of  $\sqrt{q}$  appearing the right-hand side, and hence numerical prefactors will be different if we use (2.12) instead of (2.7) as the defining characterization of the Stieltjes-Wigert weight function  $w(u)$ .

Thus the explicit  $q$ -deformed expression that satisfies (2.12) is given by

$$(2.13) \quad w(u) = E_{q^{-1}}(qu) e_q(-(qu)^{-1}) .$$

An elementary manipulation shows  $E_{q^{-1}}(u) = e_q(u)$ , and so from (2.10) the weak-coupling limit  $q \rightarrow 1$  is given by

$$(2.14) \quad \lim_{q \rightarrow 1} w(u) = \exp\left(-\frac{u}{1-q}\right) \exp\left(-\frac{u^{-1}}{1-q}\right) = \exp\left(-\frac{1}{1-q} (u + u^{-1})\right) .$$

In the unitary Chern-Simons matrix model, we substitute  $u = e^{i\theta}$  with  $0 \leq \theta \leq 2\pi$  to get

$$(2.15) \quad \lim_{q \rightarrow 1} w(u = e^{i\theta}) = \exp\left(-\frac{2}{1-q} \cos \theta\right) ,$$

and we arrive at the Gross-Witten model (1.7) with  $\alpha \propto \frac{1}{1-q}$ . Since  $q \rightarrow 1$  ( $g_s \rightarrow 0$ ) we can write  $\alpha \simeq \frac{1}{g_s}$ . The factor of 2 in (2.15) is not accurate since we have used (2.12) instead of (2.7).

The same result is contained in the  $q \rightarrow 1$  limit of the  $q$ -Hermite polynomials studied by Ismail, Stanton and Viennot in [42]. They consider polynomials with the same orthogonality properties as the ones that solve the Chern-Simons matrix model (see the discussion of the  $q \rightarrow 0$  limit below), but then apply a rescaling of the variable  $x \rightarrow (\sqrt{1-q}/2)x$  in order to obtain the correct  $q \rightarrow 1$  limit in their weight function  $v(x, q)$  as mentioned above. If we undo this rescaling and take into account that with their definitions  $x = \cos(\theta/2)$ , then their result

$$(2.16) \quad \lim_{q \rightarrow 1} v(x, q) = \exp(-x^2/2)$$

leads in our case to

$$(2.17) \quad \lim_{q \rightarrow 1} v(x, q) = \exp\left(-\frac{2}{1-q} \cos^2 \frac{\theta}{2}\right) = e^{-1/(1-q)} \exp\left(-\frac{\cos \theta}{1-q}\right) .$$

**2.3. Strong-coupling limit.** Let us now study the connection between the Chern-Simons matrix model and the Gross-Witten model in the strong-coupling limit  $g_s \rightarrow \infty$  ( $q = e^{-g_s} \rightarrow 0$ ). Recall first that both the Stieltjes-Wigert polynomials, defined on the positive real half-line, and the Rogers-Szegő polynomials, defined on the unit circle [43], solve the matrix models that appear in Chern-Simons theory [14, 15]. In the Hermitian case this was demonstrated in [14], while the unitary matrix model was proposed in [16] (but without consideration of the associated Rogers-Szegő polynomials). The explicit connection between the two models was demonstrated in [15]. The direct relationship between the two systems of orthogonal polynomials is well-known [44]. Therefore, the partition function for Chern-Simons gauge theory on  $S^3$  with gauge group  $U(N)$  can be represented both as the Hermitian matrix model

$$(2.18) \quad Z_{N,k}(S^3) = \int [dM] \exp\left(-\frac{1}{2g_s} \text{Tr}(\log M)^2\right) ,$$

with  $[dM]$  the natural invariant measure for integration over the space of  $N \times N$  Hermitian matrices, and as the unitary matrix model [16]

$$(2.19) \quad Z_{N,k}(S^3) = \int_{U(N)} dU \det \Theta_{00}(U|q) ,$$

where the theta-functions  $\Theta_{00}(e^{i\theta_i}|q)$  are defined on eigenvalues of unitary matrices  $U$  by (2.3).

The Rogers-Szegő polynomials are defined by [43]

$$(2.20) \quad H_n(z|q) := \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q z^j , \quad \begin{bmatrix} n \\ j \end{bmatrix}_q := \frac{(q)_n}{(q)_j (q)_{n-j}} .$$

They satisfy an orthogonality relation on the unit circle given by

$$(2.21) \quad \oint_{|z|=1} \frac{dz}{2\pi i z} H_m(-q^{-1/2} \bar{z} | q) H_n(-q^{-1/2} z | q) \Theta_{00}(z|q) = \frac{(q)_m}{q^m} \delta_{m,n} .$$

The orthogonality coefficients  $h_m = \frac{(q)_m}{q^m}$  are those of the Stieltjes-Wigert polynomials, that lead to the Chern-Simons partition function [14].<sup>1</sup>

The weight function appearing in (2.21) is  $\omega(z) = \Theta_{00}(z|q)$  which can be expanded in powers of  $q$  and  $z$  using (2.3). Since here  $z = e^{i\theta}$  lives on the unit circle, the expansion reads

$$(2.22) \quad \omega(\cos \theta) = 1 + 2\sqrt{q} \cos \theta + 2 \sum_{n \geq 2} q^{n^2/2} \cos n \theta .$$

The first two terms in (2.22) give the first order approximation of the weight function for the Chern-Simons matrix model as  $q \rightarrow 0$ . This result is very different from the  $q \rightarrow 1$  behaviour. The leading constant term in (2.22) corresponds to Dyson's circular ensemble [2]. The first order correction for small  $q$  is given by the weight function of the Gross-Witten model (1.7) with  $\alpha = \sqrt{q} = e^{-g_s/2} \rightarrow 0$ . Corrections to this term are given by higher powers of  $\cos \theta$ .

These behaviours are similar to those obtained in [45] for a unitary matrix model with a one-parameter family of weights which defines a  $q$ -deformation of the circular ensemble. The  $a = 0$  member of this family is the weight function [45, eq. (3)]

$$(2.23) \quad v_0(\theta, q) = \left| (\sqrt{q} e^{i\theta}; q)_\infty \right|^2 .$$

In the limit  $q \rightarrow 0$  it tends to Dyson's circular ensemble and in the limit  $q \rightarrow 1$  to the Gross-Witten model. This weight is very similar to that given by the theta-function above (although the two infinite product expansions are different).

**2.4. Comparison with multicritical and noncommutative deformations.** If one keeps only a finite number of terms in the expansion (2.22), then the corresponding matrix model is the multicritical polynomial generalization of the Gross-Witten model discussed in [24]. These results have been generalized and unitary matrix models of the form

$$(2.24) \quad Z_N^{\text{mult}}(t) = \int_{U(N)} dU \exp \left( \sum_{j=-\infty}^{\infty} t_j \text{Tr} U^j \right)$$

are of interest in various areas of field theory (see [46] for a recent review). In general, the coupling parameters  $t_j$  are restricted by solutions of Virasoro constraints on the partition function  $Z_N^{\text{mult}}(t)$ . However, if one chooses them to be the coefficients of a  $q$ -series, and in particular those of the theta-function in (2.22), then one obtains the Chern-Simons matrix model.

Our realization of the unitary matrix model describing Chern-Simons theory as a certain  $q$ -deformation of the Gross-Witten model is also reminiscent of what occurs in two-dimensional  $U(N)$  Yang-Mills theory on a noncommutative torus. In [29] it is shown that the weak-coupling limit of the partition function in this case can be regarded as coming from a modification of ordinary gauge theory by the addition of infinitely many higher Casimir operators to the action. The addition of higher Casimir operators to ordinary, two-dimensional Yang-Mills theory leads to generalized Yang-Mills theories [4].

The relationship between these noncommutative deformations and our  $q$ -deformations is most transparent in the combinatorial quantization of the gauge theory described in [30], which yields a discrete family of unitary matrix models parametrized by  $N$ -th roots of unity  $\zeta$  and partition functions given by [30, eq. (92)]

$$(2.25) \quad Z_{N,\zeta}^{\text{NC}}(\alpha) = \sum_R \frac{1}{\dim R} \int_{U(N)} dU \chi_R(U) \exp \left( -\alpha \text{Tr}(\zeta U + \bar{\zeta} U^\dagger) \right) ,$$

<sup>1</sup>These are the same orthogonality coefficients that were considered in [15]. The  $q^{-m}$  term differs from the coefficients of [14], but it only contributes a phase to the partition function. See [14] and the discussion of framing in Section 3.2 below for more details.

where the sum runs over all irreducible unitary representations  $R$  of the gauge group  $U(N)$  with dimension  $\dim R$  and characters  $\chi_R$ . The truncation of the sum in (2.25) to the trivial representation is independent of  $\zeta$  [28, §§5.2.1] and corresponds to the Gross-Witten reduction of the combinatorial quantization of ordinary gauge theory in two dimensions. The noncommutative gauge theory partition function thus generalizes that of ordinary Yang-Mills theory by perturbing it by a sum over non-trivial representations of the unitary group. Again, in contrast to the  $q$ -deformed gauge theory on  $S^2$ , this matrix model admits large  $N$  phase transitions and non-trivial double scaling limits which converge to the continuum (noncommutative) gauge theories [30, 31].

### 3. FROM THE SUTHERLAND MODEL TO CHERN-SIMONS THEORY

**3.1. Derivation of the Chern-Simons free energy.** Let us now establish the relationship between the Sutherland model and the Chern-Simons matrix model that we mentioned in the first section. We shall pay particular attention to the square of the ground state wavefunction (1.10) for the value  $\lambda = \frac{1}{2}$  of the Sutherland coupling, which is given by

$$(3.1) \quad P(q_1, \dots, q_N; L) = (\Psi_0(q_1, \dots, q_N; \frac{1}{2}, L))^2 = \prod_{i < j} \sin \frac{\pi(q_i - q_j)}{L},$$

and evaluate this probability density in the fixed and equally-spaced configuration  $q_i = c - i$ , with  $c \in \mathbb{R}$  an arbitrary constant. Then we easily find

$$(3.2) \quad P(c-1, \dots, c-N; L) = \prod_{i < j} \sin \frac{\pi(i-j)}{L} \\ = \prod_{j=1}^{N-1} \left( \sin \frac{\pi}{L} \sin \frac{2\pi}{L} \cdots \sin \frac{\pi(N-j)}{L} \right) = \prod_{j=1}^{N-1} \sin^{N-j} \left( \frac{\pi}{L} j \right).$$

This is proportional to the Chern-Simons partition function on  $S^3$  [3] once we identify the arbitrary length parameter  $L$  as

$$(3.3) \quad L = k + N$$

in terms of Chern-Simons parameters, the integer level  $k$  and the rank of the gauge group  $N$ . However, there is an additional factor  $(k + N)^{-N/2}$  in front of the product term in the Chern-Simons partition function [3]. Thus we require a renormalization of the Sutherland wavefunction (1.10) by multiplying it with the parameter  $L^{-N/4}$ . Then one has

$$(3.4) \quad \tilde{\Psi}_0(q_1, \dots, q_N; \frac{1}{2}, L) := \frac{1}{L^{N/4}} \prod_{i < j} \left( \sin \frac{\pi(q_i - q_j)}{L} \right)^{1/2}$$

with

$$(3.5) \quad (\tilde{\Psi}_0(1, \dots, N; \frac{1}{2}, k + N))^2 = Z_{N,k}(S^3).$$

In the sequel we will drop the tilde notation on the renormalized wavefunctions.

We could have obtained the same result directly from the ground state wavefunction (1.10) with the coupling  $\lambda = 1$ . In this case the Sutherland model corresponds to a theory of free fermions on a circle.<sup>2</sup> Furthermore, the configuration employed, with particles equally spaced on the circle, corresponds to the static classical equilibrium state of the Sutherland model. This condition is essential to preserve integrability when the model is discretized to construct

<sup>2</sup>A connection between Chern-Simons theory and free fermions at finite temperature is discussed in [47].

spin chain models (see [48] for a review). It follows that we can also write the free energy of Chern-Simons theory on  $S^3$  as

$$(3.6) \quad F_{N,k}(S^3) = \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \Psi_0(q_1, \dots, q_N; \lambda, k + N) .$$

Recall that the static classical equilibrium configuration of the Sutherland model, together with an additional spin interaction term in the Hamiltonian, is the one that leads to the Haldane-Shastry spin chain model [49, 50].<sup>3</sup> Hence the part without the internal spin degrees of freedom naturally induces the Chern-Simons partition function. However, the translationally invariant spin wavefunction [51, eq. (5)]

$$(3.7) \quad \check{\Psi}_0(s_1, \dots, s_N; \alpha, N) = \delta_{\sum_n s_n, 0} e^{i \frac{\pi}{2} \sum_{i \geq 0} (s_{2i+1} - 1)} \prod_{1 \leq m < n \leq N} \left( \sin \frac{\pi(n-m)}{N} \right)^{\alpha s_n s_m} ,$$

with  $s_n = \pm 1$  local spins and  $\alpha$  a positive real number, can be expressed in terms of hard-core boson variables. When  $N$  is even, one can identify the spin up (resp. down) states with empty (resp. occupied) hard-core boson states to write [51, eq. (6)]

$$(3.8) \quad \check{\Psi}_0^{\text{bos}}(n_1, \dots, n_{\frac{N}{2}}; \alpha, N) = e^{i\pi \sum_i n_i} \prod_{\substack{1 \leq i, j \leq \frac{N}{2} \\ n_i > n_j}} \left( \sin \frac{\pi(n_i - n_j)}{N} \right)^{4\alpha} ,$$

where  $n_i = 1, \dots, N$  denote the positions of  $\frac{N}{2}$  hard-core bosons in the periodic chain. According to our analysis above, this also yields the Chern-Simons partition function on  $S^3$ .

**3.2. Framing dependence.** The many-body quantum system with ground state wavefunction

$$(3.9) \quad \Psi_0^{\text{CS}}(q_1, \dots, q_N; g_s, L) = \sqrt{\alpha_N} \prod_{i=1}^N e^{-q_i^2/2g_s} \prod_{i < j} \sinh \frac{q_i - q_j}{2L}$$

is the one that appears in Chern-Simons theory [47], with the normalization constant  $\alpha_N$  giving the Chern-Simons partition function when  $L = 1$  [14]. It is also possible to obtain from (3.9) the Chern-Simons partition function  $Z_{N,k}(S^3)$  in the same way as above. This was already carried out in [36] in the context of Brownian motion (see also [52, Appendix B]).

The relevant density considered in [36] is given by

$$(3.10) \quad \Psi_0^{\text{Brown}}(q_1, \dots, q_N; t) = \prod_{i=1}^N e^{-q_i^2/t} \prod_{i < j} \sinh \frac{q_i - q_j}{2} ,$$

with  $t = -1/g_s$ . The Gaussian factor in (3.10) only contributes to a part of a framing factor [3, 53] when  $q_1, \dots, q_N$  are taken to be the components of the Weyl vector  $\rho$  of  $U(N)$ , which is given by

$$(3.11) \quad \rho = \sum_{i=1}^N \left( \frac{N+1}{2} - i \right) e_i$$

with  $e_i$  the standard basis of unit vectors in  $\mathbb{R}^N$ . This is simply the equilibrium configuration of the Sutherland model that we considered above. The contribution of the Gaussian factor is then the phase [36]

$$(3.12) \quad e^{-|\rho|^2/t} = \exp \left( \frac{i\pi N(N^2 - 1)}{12(k + N)} \right) .$$

<sup>3</sup>More generally, one can include internal  $SU(N)$  colour degrees of freedom for the particles.

Thus the model that leads directly to the Chern-Simons partition function with the canonical framing on  $S^3$  is the Sutherland model, while the model with ground state wavefunction (3.10) leads to the Chern-Simons partition function in the matrix model framing [36].

Generally, the contribution of a framing  $\Pi$  on a three-manifold  $M$  (here  $M = S^3$ ), i.e. a choice of trivialization of the bundle  $TM \oplus TM$ , to the partition function for gauge group  $G$  (here  $G = U(N)$ ) is parametrized by an integer  $s \in \mathbb{Z}$  and given by [53]

$$(3.13) \quad \delta(M, \Pi) = \exp\left(\frac{2\pi i s}{24} c\right) = \exp\left(\frac{2\pi i s}{24} \frac{k \dim G}{k+h}\right) = \exp\left(\frac{\pi i s |\rho|^2 k}{h(k+h)}\right),$$

where  $h$  is the dual Coxeter number of  $G$  (here  $h = N$ ). In the second equality here we have used the explicit expression for the central charge  $c$  of the associated Wess-Zumino-Witten conformal field theory based on the affine extension of  $G$ . In the third equality we have used the Freudenthal-de Vries formula to relate the dimension of the Lie group  $G$  to the length of its Weyl vector  $\rho$  (see [53] for details). Thus if we write the framing factor (3.13) in the form

$$(3.14) \quad \delta(M, \Pi) = \exp\left(\frac{\pi i s |\rho|^2}{h}\right) \exp\left(-\frac{\pi i s |\rho|^2}{k+h}\right),$$

we see that the contribution (3.12) is given by the second factor in (3.14).

A phase contribution that does not include the level  $k$  of the Chern-Simons gauge theory, as in the first factor of (3.14), appears when one goes from the hyperbolic to the trigonometric case and conversely. Indeed, one can also consider the hyperbolic Sutherland model with ground state wavefunction

$$(3.15) \quad \Psi_0^{\text{hyp}}(q_1, \dots, q_N; \lambda, L) = \prod_{i < j} \left( \sinh \frac{\pi (q_i - q_j)}{L} \right)^\lambda,$$

which coincides with that of the Chern-Simons model (3.9) at  $\lambda = 1$  and without the Gaussian factors in the wavefunction. Then one obtains a relation analogous to (3.5), with an additional phase factor due to the appearance of hyperbolic sine functions instead of trigonometric ones (this factor also arises when one uses (3.10)). Repeating the procedure in (3.2) using the relation  $\sinh(ix) = i \sin x$  we see that we now have to take  $L = -i(k+N)$  and that the additional phase factor, including the corresponding renormalization (3.4), is  $\exp\left(\frac{1}{4} i \pi N^2\right)$ , which also appears in the random matrix theory computation of  $\alpha_N$  [14].

**3.3. Extension to generic root systems.** The results we have described are not restricted to unitary gauge groups, and can be directly extended to orthogonal and symplectic gauge groups using generalizations of the Sutherland model appropriate to other semi-simple Lie algebras [54]. The generalized Calogero-Sutherland model is defined by the quantum Hamiltonian operator [54]

$$(3.16) \quad H = -\sum_{i=1}^N \frac{\partial^2}{\partial q_i^2} + \sum_{\alpha \in \Delta} \frac{g_{|\alpha|}}{\sin^2 \frac{1}{2}(q, \alpha)},$$

where  $\Delta$  is the root system of a semi-simple Lie algebra in an  $N$ -dimensional real vector space  $V$  with inner product  $(-, -)$ ,  $q = (q_1, \dots, q_N) \in V$ , and  $g_{|\alpha|} = \lambda_{|\alpha|}(\lambda_{|\alpha|} - 1) \pi^2 / L^2$  are coupling constants which depend only on the lengths  $|\alpha|$  of the root vectors  $\alpha \in \Delta$  (i.e. on the orbits of the Weyl group). The usual Sutherland model (1.9) corresponds to the  $A_{N-1}$  root system.

The  $BC_N$  root system contains three distinct Weyl group orbits and hence three coupling constants. The Hamiltonian is

$$(3.17) \quad \begin{aligned} H = & - \sum_{i=1}^N \frac{\partial^2}{\partial q_i^2} + \frac{2\lambda(\lambda-1)\pi^2}{L^2} \sum_{i<j} \left( \sin \frac{\pi(q_i - q_j)}{L} \right)^{-2} \\ & + \frac{2\lambda(\lambda-1)\pi^2}{L^2} \sum_{i<j} \left( \sin \frac{\pi(q_i + q_j)}{L} \right)^{-2} \\ & + \frac{\lambda_2(\lambda_2-1)\pi^2}{L^2} \sum_{i=1}^N \left( \sin \frac{\pi q_i}{2L} \right)^{-2} + \frac{\lambda_1(\lambda_1-1)\pi^2}{L^2} \sum_{i=1}^N \left( \sin \frac{\pi q_i}{L} \right)^{-2}, \end{aligned}$$

with corresponding ground state wavefunction

$$(3.18) \quad \begin{aligned} \Psi_0(q_1, \dots, q_N; \lambda_1, \lambda_2, \lambda, L) = & \prod_{i=1}^N \sin^{\lambda_1} \left( \frac{\pi q_i}{L} \right) \sin^{\lambda_2} \left( \frac{\pi q_i}{2L} \right) \\ & \times \prod_{i<j} \left( \sin \frac{\pi(q_i - q_j)}{2L} \right)^\lambda \left( \sin \frac{\pi(q_i + q_j)}{2L} \right)^\lambda. \end{aligned}$$

The case  $\lambda_1 = 0$  corresponds to the  $B_N$  root lattice,  $\lambda_2 = 0$  to the  $C_N$  root lattice, and  $\lambda_1 = \lambda_2 = 0$  to the  $D_N$  root lattice. This generalized Sutherland model is related, as in the case of the  $A_{N-1}$  system, to the partition function of Chern-Simons gauge theory on  $S^3$  with orthogonal and symplectic gauge groups. The generic framing dependence is described by (3.14).

For illustration, let us consider explicitly the Lie group  $Sp(2N)$ . The configuration is again fixed and equispaced as

$$(3.19) \quad (q_1, \dots, q_N) = (N, N-1, N-2, \dots, 1),$$

and this choice corresponds to the Weyl vector  $\rho$  of the Weyl chamber of  $C_N$ . Note that there is no essential difference with the  $A_{N-1}$  system treated earlier, where the equispaced configuration was taken to be  $q_i = c - i$  with arbitrary but fixed  $c \in \mathbb{R}$ . Evaluating (3.18) with  $\lambda = \lambda_1 = \frac{1}{2}$  and  $\lambda_2 = 0$  yields the probability density [36]

$$(3.20) \quad \left( \Psi_0(N, N-1, \dots, 1; \frac{1}{2}, 0, \frac{1}{2}, L) \right)^2 = \prod_{j=1}^{2N+1} \left( \sin \frac{j}{2L} \right)^{f(j)},$$

where

$$(3.21) \quad f(j) = \begin{cases} N - \frac{j}{2} - \frac{1}{2} & j \text{ odd} \leq N \\ N - \frac{j}{2} & j \text{ even} < N \\ N - \frac{j}{2} + \frac{1}{2} & j \text{ odd} > N \\ N - \frac{j}{2} + 1 & j \text{ even} > N. \end{cases}$$

With the identification (3.3), this is the partition function of Chern-Simons theory on  $S^3$  with gauge group  $Sp(2N)$  [55].

**3.4. Derivation of Wilson line observables.** The Sutherland model also delivers other quantum topological invariants, besides partition functions, in a similar way. For example, quantum dimensions [56] can be obtained in this way as well. In Chern-Simons gauge theory, the Wilson line invariants associated to the unknot give rise to the quantum dimensions which reduce, in the semiclassical limit  $k \rightarrow \infty$  ( $g_s \rightarrow 0$ ), to the dimensions of representations of the gauge group. For example, the irreducible representations  $R$  of the gauge group  $SU(N)$  can be parametrized by the lengths of the rows of Young tableaux  $\mu_i$ ,  $i = 1, \dots, N$ , with  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_N$ . Using

again (3.4) and (3.2), but now with these weights as the positions  $q_i = \mu_i$  of the particles, i.e. an inhomogeneous configuration, one obtains

$$(3.22) \quad \dim_q R = W_R(\text{unknot}) = \frac{1}{L^{N/2}} \prod_{i < j} \sin \frac{\pi(\mu_i - \mu_j)}{L}$$

again with the identification (3.3). In this way knot and link invariants in Chern-Simons gauge theory correspond to non-equilibrium configurations of particle positions in the Sutherland model. It would be interesting to study excited states of the Sutherland model in its classical equilibrium configuration to see if they give rise to quantum dimensions or other observables of Chern-Simons theory.

**3.5. Hamiltonian analysis.** Recall that the hyperbolic Sutherland model and the Chern-Simons model differ in a Gaussian factor at the level of the ground state wavefunction (3.9). In the relationship with Chern-Simons theory, this is only a (framing) phase contribution to the partition function. At the level of the corresponding Hamiltonians, they are related by [47]

$$(3.23) \quad H_{\text{CS}} = H_{\text{hyp}} + \frac{1}{g_s L} \sum_{i < j} (q_i - q_j) \coth \left( \frac{q_i - q_j}{L} \right)$$

where  $H_{\text{hyp}}$  is the Hamiltonian of the hyperbolic Sutherland model. The reason for the appearance of a two-body term at the Hamiltonian level to explain a one-body factor in the ground state wavefunction is explained in [47] (together with some other properties of the model).

At large distances, the two-body potential is given by a one-dimensional Coulomb potential  $V(x - x') = |x - x'|$ . Thus it has strongly confining properties, and while both the Sutherland model and the Chern-Simons model are connected to Chern-Simons gauge theory, the addition of such a potential leads to a very different correspondence. For example, in [57] the equivalent relationship between the Sutherland model and the Thirring model is presented, and a generalized Sutherland model with a two-body potential is also studied. It is shown there that in order for the correspondence between the Sutherland and Luttinger-Thirring models to hold, the generic two-body potential  $V(x)$  cannot decay slower than  $|x|^{-2}$  for large  $x$ .<sup>4</sup> As explained in [57], the one-dimensional Coulomb potential has Fourier transform of the form  $\tilde{V}(k) = A/k^2$  and the physics is that of one-dimensional quantum electrodynamics, rather than the Luttinger-Thirring behaviour of the Sutherland model. Indeed, a linear confining potential (such as the one-dimensional Coulomb potential) is one of the main ingredients of a massive Schwinger model. It would be interesting to understand further the implications of the fact that the two-body potential of the Chern-Simons model is, at large separations, a one-dimensional Coulomb potential. The exactly solvable Chern-Simons model is also related to other systems in condensed matter physics [33], such as Laughlin and multilayer wavefunctions on thin cylinders, and also through a connection between the Sutherland model and Luttinger liquids with boundaries.

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<sup>4</sup>This ensures that the Fourier transform of the potential is not of the form  $\tilde{V}(k) = A/k^\sigma$  with  $\sigma > 1$  (see [57] for details).

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