SCHUR POLYNOMIALS AND BIORTHOGONAL RANDOM MATRIX ENSEMBLES

MIGUEL TIERZ

Abstract. The study of the average of Schur polynomials over a Stieltjes-Wigert ensemble has been carried out in [J. Math. Phys. **48**, 023507 (2007)] (arXiv:hep-th/0609167), where it was shown that it is equal to quantum dimensions. Using the same approach, we extend the result to the biorthogonal case. We also study, using the Littlewood–Richardson rule, some particular cases of the quantum dimensions result. Finally, we show that the notion of Giambelli compatibility of Schur averages, introduced in [Adv. Appl. Math. **37**, 209 (2006)] (arXiv:math-ph/0501123), also holds in the biorthogonal setting.

1. Introduction

The study of random matrix ensembles is a subject of considerable mathematical interest, with applications in different areas of physics [1, 2, 3]. The definition of the joint probability distribution $P(M)$ of the matrix elements of a N by N matrix M is [1]

$$P(M) = C_N \exp[-\mathrm{Tr}V(M)] \ , \tag{1.1}$$

with an arbitrary $V(M)$, provided existence of the partition function C_N^{-1}. The integration of (1.1) over the parameters related to the eigenvectors, leads to the well-known joint probability distribution of the eigenvalues [1]

$$P(x_1, ..., x_N) = C_N \prod_{i<j} |x_i - x_j|^\beta \prod_{i=1}^{N} \exp[-V(x_i)] \ . \tag{1.2}$$

We can see in (1.2) that level repulsion described by the Vandermonde determinant is originated from the Jacobian, that appears when passing from the integration over independent elements of the matrix ensemble to the integration over the smaller space of its N eigenvalues. The parameter β, with values $1, 2$ or 4 describe the symmetry of the ensemble (orthogonal, unitary and symplectic, respectively). In [4], a one parameter generalization of the orthogonal polynomial ensembles [1, 5] is studied. The joint probability densities of these ensembles have the form

$$P(x_1, \ldots, x_N) = C_N \prod_{i=1}^{N} \omega(x_i) \prod_{i<j} \left[(x_i - x_j)(x_i^\theta - x_j^\theta)\right] , \tag{1.3}$$

where θ is a fixed positive number. These are biorthogonal ensembles and the usual orthogonal polynomial ensembles correspond to $\theta = 1$. In [4], the classical cases, corresponding to Hermite, Laguerre and Jacobi polynomials were studied.

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In analogy with the usual Hermitian case ($\theta = 1$) [1], one can study (1.3) by considering a pair of biorthogonal polynomials

$$\int \omega\left(x\right) Y_n\left(x,k\right) Z_m\left(x,k\right) dx = h_{n,k}\delta_{n,m}, \tag{1.4}$$

with

$$\begin{aligned} \int Y_n\left(x,k\right) x^{\theta j} \omega\left(x\right) dx &= \alpha_n^{(\theta)} \delta_{n,j}, \\ \int Z_n\left(x,k\right) x^{j} \omega\left(x\right) dx &= \beta_n^{(\theta)} \delta_{n,j}. \end{aligned} \tag{1.5}$$

In [6], we studied the biorthogonal ensemble characterized by a log-normal weight $\omega\left(x\right) = \mathrm{e}^{-\kappa^2 \log^2 x_i}$. The orthogonal polynomials associated to this weight are the Stieltjes-Wigert polynomials [7]. We were motivated by the appearance of the Stieltjes-Wigert random matrix model in Chern-Simons theory [8, 9]. The model is

$$Z^{P,Q} = \int \prod_{i=1}^{N} \mathrm{e}^{-u_i^2/2g_s} \prod_{i<j} \left(2\sinh\frac{u_i-u_j}{2P}\right)\left(2\sinh\frac{u_i-u_j}{2Q}\right)\frac{du_i}{2\pi}. \tag{1.6}$$

Note that (1.3) is exactly the type of ensemble that (1.6) leads to, since

$$\begin{aligned} Z^{P,Q} &= \int \prod_i \frac{\mathrm{d}u_i}{2\pi} \mathrm{e}^{-u_i^2/2g_s} \prod_{i<j} (2\sinh(\frac{u_i-u_j}{2P}))(2\sinh(\frac{u_i-u_j}{2Q}) \\ &= q^{-\frac{N\alpha^2}{2}} \int \prod_i \frac{\mathrm{d}y_i}{2\pi} \mathrm{e}^{-\kappa^2\log^2 y_i} \prod_{i<j} (y_i^{1/P} - y_j^{1/P})(y_i^{1/Q} - y_j^{1/Q}), \end{aligned} \tag{1.7}$$

with $u_i = \log \mathrm{e}^{\frac{\alpha}{2\kappa^2}} y_i$, $\kappa^2 = \frac{1}{2g_s}$ and $\alpha = -1 - \frac{(P+Q)(N-1)}{2PQ}$. Finally, with $y_i = \mathrm{e}^{\frac{P-1}{2\kappa^2 P}} x_i^P$ and some rewriting

$$Z^{P,Q} = P^N \mathrm{e}^{-\frac{N}{4\kappa^2}(\frac{1}{P} + \frac{(P+Q)(N-1)}{2PQ})^2} \int \prod_i \frac{\mathrm{d}x_i}{2\pi} \mathrm{e}^{-\kappa^2 P^2 \log^2 x_i} \prod_{i<j} (x_i - x_j)(x_i^{P/Q} - x_j^{P/Q}),$$

which is of the form (1.3) with the log-normal (Stieltjes-Wigert) weight function $\omega\left(x\right) = \mathrm{e}^{-\kappa^2 P^2 \log^2 x_i}$. The corresponding q-parameter is then $q = \mathrm{e}^{-\frac{1}{2\kappa^2 P^2}} = \mathrm{e}^{-\frac{g_s}{P^2}}$. In [6] we mainly studied two problems:

(1) The biorthogonal ensemble itself: the construction of the Stieltjes-Wigert biorthogonal polynomials and the computation of $Z^{P,Q}$.
(2) The ordinary Stieltjes-Wigert Hermitian ensemble ($P = Q = 1$ above) with the insertion of a Schur polynomial.

Recall that Schur polynomials $\mathfrak{s}_\lambda$ [12, 13] constitute a basis of symmetric functions in a given set of variables $x = (x_i)$ and are indexed by Young diagrams λ. In particular, we showed

$$\begin{aligned} < \mathfrak{s}_\lambda(M) >_w &= \int [dM] \mathfrak{s}_\lambda(M) \mathrm{e}^{-\frac{1}{2g_s}\mathrm{Tr}(\log M)^2} \\ &= q^{-n|\lambda| - \frac{1}{2}C_\lambda^{U(n)}} \mathcal{D}_\lambda, \end{aligned} \tag{1.8}$$

where $C_\lambda^{U(n)}$ is the Casimir of $U(N)$ and the last term are the quantum dimensions, defined by the q-hook formula [11]

$$\mathcal{D}_\lambda \equiv \prod_{x\in\lambda} \frac{\lfloor n+c(x)\rfloor}{\lfloor h(x)\rfloor}, \tag{1.9}$$

where $\lfloor n\rfloor = \left(q^{n/2}-q^{-n/2}\right)/\left(q^{1/2}-q^{-1/2}\right)$ denotes the q-number and for each box $x=(i,j)$ of the diagram $h(x)\equiv \lambda_i+\lambda_j'-i-j+1$ is the hook-length and $c(x)\equiv j-i$ the content of x. If the variables x are seen as eigenvalues of some matrix $M\in sl_n$ then $\mathfrak{s}_\lambda(M)\equiv \mathrm{Tr}_\lambda(M)$ is the trace of M in the representation associated to λ. The Casimir of the representation labeled by the Young diagram λ and $|\lambda|$ its total number of boxes is

$$C_\lambda^{U(n)} = (n+1)|\lambda| + \sum_i (\lambda_i^2 - 2i\lambda_i). \tag{1.10}$$

In the following Section, we study the problem that results from combining (1) and (2) above. That is to say, the computation of the average of a Schur polynomial over a biorthogonal ensemble (1.3) with a log-normal weight $\omega(x)=\mathrm{e}^{-\kappa^2\log^2 x_i}$. In addition to [6], other random matrix averages of Schur polynomials have also been studied in [14, 15, 16, 17, 18, 19], with different techniques and motivations. We will also analyze simple particular cases of the quantum dimensions result in [6].

In Section 4, we will study a basic property of a biorthogonal ensemble with a generic weight function and a Schur polynomial. More precisely, a special property of an orthogonal polynomial ensemble with a Schur polynomial is the Giambelli compatibility, introduced and analyzed in detail in [10]. We shall show that the same property holds if we deal with biorthogonal ensembles. We now briefly introduce the notion of Giambelli compatibility. The partitions λ that parametrize Schur functions can be written in Frobenius notation (see [20] for a simple explanation of this notation):

$$\lambda = (p_1,\ldots,p_d|q_1,\ldots,q_d). \tag{1.11}$$

The Schur functions satisfy a basic identity, called the Giambelli formula

$$\mathfrak{s}_{(p_1,\ldots,p_d|q_1,\ldots,q_d)} = \det\left[\mathfrak{s}_{(p_i|q_j)}\right]_{i,j=1}^d. \tag{1.12}$$

In [10], it is shown that the Giambelli formula remains invariant under the averaging of the Schur function over a generic orthogonal polynomial ensemble

$$\left\langle \mathfrak{s}_{(p_1,\ldots,p_d|q_1,\ldots,q_d)}\right\rangle = \det\left[\left\langle\mathfrak{s}_{(p_i|q_j)}\right\rangle\right]_{i,j=1}^d. \tag{1.13}$$

Since we are studying a biorthogonal ensemble, we shall show in Section 4 that the method of [10] extends in a straightforward way to the biorthogonal case and hence, the Giambelli compatibility property also holds for such an ensemble. In contrast to Section 2 and Section 3, the discussion considers a generic weight and does not focus on the Stieltjes-Wigert model.

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2. Schur average in the Stieltjes-Wigert biorthogonal ensemble

The computation of random matrix ensembles with a Schur insertion is not a well developed topic. It is interesting from a mathematical point of view as, presumably, one needs to combine results from algebraic combinatorics with the standard orthogonal polynomial technique in random matrix theory. Works studying Schur averages over random matrix ensembles are [14]-[19].

2.1. Characteristic polynomials in the biorthogonal case.

In the orthogonal case [6], recall that the average of the following polynomial

$$P_k(M) \equiv \prod_{i=1}^{k} \det(x_i - M) \tag{2.1}$$

is obtained using a result by Brézin and Hikami [22]

$$< P_k(x;M) >_w = \frac{1}{\mathfrak{a}_\delta(x)} \begin{vmatrix} \pi_n(x_1) & \dots & \pi_{n+k-1}(x_1) \\ \vdots & \vdots\vdots & \\ \pi_n(x_k) & \dots & \pi_{n+k-1}(x_k) \end{vmatrix}, \tag{2.2}$$

where $\mathfrak{a}_\delta(x)$ is the Vandermonde determinant[1] and $\pi_n(x)$ are the monic orthogonal polynomials. We want to write an analogous formula in the biorthogonal case. Following the steps in [22], it is easy to show that

$$< P_k(x;M) >_{w,\theta} = \frac{1}{\mathfrak{a}_\delta(x)} \begin{vmatrix} \rho_n(x_1,\theta) & \dots & \rho_{n+k-1}(x_1,\theta) \\ \vdots & \vdots\vdots & \\ \rho_n(x_k,\theta) & \dots & \rho_{n+k-1}(x_k,\theta) \end{vmatrix}, \tag{2.3}$$

$$< P_k(x^\theta;M^\theta) >_{w,\theta} = \frac{1}{\mathfrak{a}_\delta(x^\theta)} \begin{vmatrix} \tau_n(x_1,\theta) & \dots & \tau_{n+k-1}(x_1,\theta) \\ \vdots & \vdots\vdots & \\ \tau_n(x_k,\theta) & \dots & \tau_{n+k-1}(x_k,\theta) \end{vmatrix}. \tag{2.4}$$

2.2. Extension of the quantum dimensions result to the biorthogonal case.

We first recall here the definition of the monic Stieltjes-Wigert polynomials

$$\pi_n(x;\theta|q) = \sum_{j=0}^{n} (-1)^n q^{(j-n)(j+n+\frac{1}{2})} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j. \tag{2.5}$$

From our previous work [6], the monic biorthogonal Stieltjes-Wigert polynomials read

$$\rho_n(x;\theta|q) = \sum_{j=0}^{n} (-1)^n q^{\frac{1}{2}(j-n)((1+\theta)(j+n)+2-\theta)} \begin{bmatrix} n \\ j \end{bmatrix}_{q^\theta} x^j, \tag{2.6}$$

and

$$\tau_n(x;\theta|q) = \sum_{j=0}^{n} (-1)^n q^{\frac{\theta}{2}(j-n)((1+\theta)(j+n)+1)} \begin{bmatrix} n \\ j \end{bmatrix}_{q^\theta} x^{\theta j}. \tag{2.7}$$

[1] We adopt the notations of [12]

Now, let us turn to the computation of the quantum dimensions as performed in [6]. First, recall that the polynomial defined in (2.1) generates the Schur polynomials in the following way

$$\prod_{i=1}^{k} \det(x_i - M) = \sum_{\lambda\,;\,\lambda_1\leq k} (-1)^{|\lambda|}\mathfrak{s}_\lambda(M)\mathfrak{s}_{\tilde{\lambda}'}(x). \tag{2.8}$$

We thus have to evaluate the determinant on the r.h.s. of (2.2), this time in terms of the ρ_n polynomials, rather than the π_n ones that we employed in [6]. That is, we shall employ (2.3) instead of (2.2).

2.2.1. *1 column case.* One can read off the result for representations associated to 1-column Young diagrams from the polynomials themselves

$$< \mathfrak{s}_{(1^j)}(M) >_{w,\theta} = q^{-\frac{j}{2}((1+\theta)(2n-j)+2-\theta)} \begin{bmatrix} n \\ j \end{bmatrix}_{q^\theta} = q^{-\frac{j}{2}((2\theta+1)n-j+2-\theta)} \begin{bmatrix} n \\ j \end{bmatrix}_{q^\theta}. \tag{2.9}$$

As in [6], we use $\lfloor n \rfloor_q = q^{(1-n)/2}\,[n]_q$. Notice that, in this case, it is not clear how to give a group theoretical interpretation of the exponent in the prefactor, in contrast to the case when $\theta = 1$ [6] (see (2.15) below).

2.2.2. *General case.* We have to consider

$$\begin{aligned} \mathfrak{a}_\delta(x)\Delta \;=\; & \sum_{i_1,\ldots,i_k} \sum_{\sigma\in\mathfrak{S}_k} \epsilon\,(\sigma)\,(-1)^{i_1+\ldots+i_k} \prod_{j=1}^{k} q^{-\frac{i_j}{2}((1+\theta)(2n+2\sigma(j)-2-i_j)+2-\theta)} \\ & \times \prod_{j=1}^{k} \begin{bmatrix} n+\sigma\,(j)-1 \\ i_j \end{bmatrix}_{q^\theta} x_j^{n+\sigma(j)-1-i_j}. \end{aligned} \tag{2.10}$$

Performing the relabelling $i_j \to i_j + \sigma(j) - 1$, we obtain

$$\begin{aligned} & \sum_{i_1,\ldots,i_k} (-1)^{i_1+\ldots+i_k+\frac{k(k-1)}{2}} \prod_{j=1}^{k} q^{-\frac{i_j}{2}[(1+\theta)(2n-i_j)+2-\theta]-\frac{(j-1)}{2}[(1+\theta)(2n+j)+1-2\theta]} \\ & \qquad \times \left(\sum_{\sigma\in\mathfrak{S}_k} \epsilon(\sigma) \prod_{j=1}^{k} \begin{bmatrix} n+\sigma\,(j)-1 \\ i_j+\sigma\,(j)-1 \end{bmatrix}_{q^\theta} \right) \prod_{j=1}^{k} x_j^{n-i_j}, \end{aligned} \tag{2.11}$$

or more simply

$$\begin{aligned} & \sum_{i_1,\ldots,i_k} (-1)^{i_1+\ldots+i_k+\frac{k(k-1)}{2}} \prod_{j=1}^{k} q^{-\frac{i_j}{2}[(1+\theta)(2n-i_j)+2-\theta]-\frac{(j-1)}{2}[(1+\theta)(2n+j)+1-2\theta]} \\ & \qquad \times \det_{1\leq a,b\leq k} \left(\begin{bmatrix} n+b-1 \\ i_a+b-1 \end{bmatrix}_{q^\theta} \right) \prod_{j=1}^{k} x_j^{n-i_j}. \end{aligned} \tag{2.12}$$

Now, in [6] we showed that

$$\det_{1\leq a,b\leq k} \left(\begin{bmatrix} n+b-1 \\ i_a+b-1 \end{bmatrix}_{q^\theta} \right) = q^{\theta n(\lambda')} \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q^\theta} = q^{\theta[n(\lambda')-n(\lambda)+\frac{1}{2}(n-1)|\lambda|]}\mathcal{D}^\theta_\lambda, \tag{2.13}$$

with λ' the partition conjugate to λ and equal to $(i_1, i_2+1, \ldots, i_k+k-1)$. $\mathcal{D}^\theta_\lambda$ is the quantum dimension of the representation, but in terms of the quantum parameter q^θ. Therefore, we eventually obtain the expression of the average of a Schur polynomial as

$$\boxed{< \mathfrak{s}_\lambda(M) >_{w,\theta} = q^{f(\lambda,\theta)}\mathcal{D}_\lambda^\theta,} \tag{2.14}$$

with

$$\begin{aligned} f(\lambda,\theta) =& \theta\left[n(\lambda') - n(\lambda) + \frac{1}{2}(n-1)|\lambda|\right] \\ & - \sum_j \frac{i_j}{2}\left[(1+\theta)(2n-i_j)+2-\theta\right] + \frac{(j-1)}{2}\left[(1+\theta)(2n+j)+1-2\theta\right]. \end{aligned}$$

When $\theta = 1$, it reduces to

$$f(\lambda,1) = -\frac{1}{2}\left((3n+1)|\lambda| + \sum_i \lambda_i^2 - 2i\lambda_i\right) = -n|\lambda| - \frac{1}{2}C_\lambda^{U(n)}. \tag{2.15}$$

For general θ, it is not obvious how to give the exponent factor a group theoretical interpretation.

3. Quantum dimensions: particular cases

In the previous Section, we have extended the quantum dimensions result of [6] to the biorthogonal setting. In both computations, we have employed a rather general approach, valid for any representation. However, it is also interesting to deal with more elementary, particular cases, involving simple representations, in order to improve our understanding through elementary but explicit results. In addition, important results like the celebrated product formula for the Schur polynomials in terms of the Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$ [21] are employed in the process, and some sort of q-Littlewood-Richardson coefficients are found. In sake of simplicity, we focus on the orthogonal ($\theta = 1$) case of [6].

3.1. **One column case.** Using the same notation as in [6], the quantum dimension of the j-th fundamental representation of A_{n-1}, which is associated to the partition (1^j), or a one-column Young tableau of length j, is

$$\mathcal{D}_{(1^j)} \equiv \dim_q \Lambda_{(j)} = \begin{bmatrix} n \\ j \end{bmatrix}_q . \tag{3.1}$$

The monic Stieltjes-Wigert polynomials can be written

$$\pi_n(x) = \sum_{j=0}^n (-1)^{n-j} q^{(j-n)(j+n+\frac{1}{2})} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j = < \det(x-M) >_w . \tag{3.2}$$

In addition, the following formula holds for the characteristic polynomial

$$\det(x-M) = \sum_{j=0}^n (-1)^{n-j} \mathfrak{s}_{(1^{n-j})}(M) x^j, \tag{3.3}$$

with $\mathfrak{s}_\lambda(M)$ the Schur polynomial associated to the partition λ. Therefore

$$\sum_{j=0}^n (-1)^{n-j} < \mathfrak{s}_{(1^{n-j})}(M) > x^j = \sum_{j=0}^n (-1)^{n-j} q^{(j-n)(j+n+\frac{1}{2})} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j, \tag{3.4}$$

from which we obtain

$$< \mathfrak{s}_{(1^j)}(M) > = q^{-j(2n-j+\frac{1}{2})} \begin{bmatrix} n \\ j \end{bmatrix}_q = q^{-\frac{j}{2}(3n-j+1)} \begin{bmatrix} n \\ j \end{bmatrix}_q . \tag{3.5}$$

For later convenience, we also write $< \mathfrak{s}_{(1^{n-j})}(M) >= q^{(j-n)(n+\frac{j+1}{2})}\mathcal{D}_{(1^{n-j})}$. In particular we have

$$< \mathfrak{s}_{\square} >= q^{-\frac{3n}{2}}\mathcal{D}_{\square}. \tag{3.6}$$

As a check that Eq. (3.5) is correct, note that the exponent $\frac{j}{2}(3n-j+1)$ can be rewritten as

$$\frac{j}{2}(3n-j+1) = l_{(1^j)}n + \frac{1}{2}C^{U(n)}_{(1^j)}, \tag{3.7}$$

where $l_{(1^j)}$ is the box number of the tableau and $C^{U(n)}_{\lambda}$ is the Casimir of the representation of $U(n)$ associated to the partition λ

$$C^{U(n)}_{\lambda} = l(\lambda)(n+1) + \sum_i (\lambda_i^2 - 2i\lambda_i^2), \tag{3.8}$$

with $l(\lambda)$ the number of boxes of the diagram. In the case of the j-th fundamental representation the Casimir is, as expected: $C^{U(n)}_{(1^j)} = nj - j^2 + j$.

3.2. **Some simple two-columns.** We can examine in detail other representations, like $< \mathfrak{s}_{\square\square}(M) >$. We employ, as in the general method, the relation established by Brézin and Hikami in [22], for the product of characteristic polynomials. We only need a second moment

$$< \det(x-M)^2 >= \begin{vmatrix} \pi_n & \pi_{n+1} \\ \pi'_n & \pi'_{n+1} \end{vmatrix}. \tag{3.9}$$

Expanding the highest degree terms on the l.h.s., we see that

$$< \det(x-M)^2 > = x^{2n} - 2 < \mathfrak{s}_{\square} > x^{2n-1} + (3 < \mathfrak{s}_{\substack{\square\\\square}} > + < \mathfrak{s}_{\square\square} >)x^{2n-2} -$$
$$- (4 < \mathfrak{s}_{\substack{\square\\\square\\\square}} > +2 < \mathfrak{s}_{\substack{\square\square\\\square}} >)x^{2n-3} + \dots,$$

where we have used the relevant particular cases of the product formula for the Schur polynomials in terms of the Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$ [21, 12]

$$\mathfrak{s}_{\lambda}\mathfrak{s}_{\mu} = \sum_{\nu} c^{\nu}_{\lambda\mu}\mathfrak{s}_{\nu}, \tag{3.10}$$

for arbitrary partitions λ, μ and ν. Now, a straightforward matching with the r.h.s. gives

$$< \mathfrak{s}_{\square\square} >= q^{-4n}\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q = q^{-3n-1}\mathcal{D}_{\square\square}. \tag{3.11}$$

Normalizing the Schur operators through $\bar{\mathfrak{s}}_{\square} \equiv q^{\frac{3n}{2}}\mathfrak{s}_{\square}$ so that $< \bar{\mathfrak{s}}_{\square} >= \mathcal{D}_{\square}$, then we have

$$< \bar{\mathfrak{s}}_{\square} >^2 = \mathcal{D}_{\substack{\square\\\square}} + \mathcal{D}_{\square\square} \quad \text{and} \quad < \bar{\mathfrak{s}}^2_{\square} >= q\mathcal{D}_{\substack{\square\\\square}} + q^{-1}\mathcal{D}_{\square\square}. \tag{3.12}$$

This somehow suggests a definition of q-Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}(q)$. For instance, from this particular case, we have

$$c^{\substack{\square\\\square\\\square}}_{\square,\square}(q) = q \text{ and } c^{\substack{\square\square\\\square}}_{\square,\square}(q) = q^{-1}. \tag{3.13}$$

Note that if we define, more generally, for any partition λ

$$\bar{\mathfrak{s}}_\lambda \equiv q^{\frac{3nl(\lambda)}{2}}\mathfrak{s}_\lambda, \tag{3.14}$$

we have the following formula for the fundamental representations

$$\bar{\mathfrak{s}}_{(1^j)} = q^{\frac{j(j-1)}{2}}\mathcal{D}_{(1^j)}, \tag{3.15}$$

or equivalently

$$c^{\Lambda_{j+1}}_{\Lambda_j,\square} = q^{\frac{j(j+1)}{2}}. \tag{3.16}$$

Analyzing the degree $(2n-3)$ part of Eq. 3.9 we see that

$$<\mathfrak{s}_{(2,1)}> = q^{-6n+\frac{5}{2}}\frac{[n+1][n][n-1]}{[3]} = q^{-\frac{9n}{2}}\mathcal{D}_{(2,1)} \tag{3.17}$$

because

$$\mathcal{D}_{(2,1)} = \frac{\lfloor n+1\rfloor\lfloor n\rfloor\lfloor n-1\rfloor}{\lfloor 3\rfloor}, \tag{3.18}$$

by the q-Hook formula (1.9) [11]. Once again, we see that the relation

$$<\mathfrak{s}_\lambda> = q^{-(nl(\lambda)+\frac{1}{2}C^{U(n)}_\lambda)}\mathcal{D}_\lambda \tag{3.19}$$

holds, as expected.

We have thus studied so far particular cases of the general result proved in [6] and extended to the biorthogonal case. In contrast to this latter detailed study of particular cases, in the next Section we show that, with a generic weight function, the Giambelli compatibility property (1.13) extends to the biorthogonal setting.

4. Giambelli compatibility

We have briefly presented in the Introduction the Giambelli compatibility property, introduced and studied in detail in [10]. We shall see now that the argument that leads to show that the unitary ensemble of random matrix theory is Giambelli compatible in the sense of [10], extends to the biorthogonal case. The proof is strictly the same (we reproduce it for clarity and to fix the notation), because the procedure used in [10] applies also to the biorthogonal ensemble. Calling λ an arbitrary Young diagram and $\mathfrak{s}_\lambda(x)$ its associated Schur polynomials, one has

$$\mathfrak{s}_\lambda(x) = \frac{\det(x_i^{\lambda_j+N-j})}{\det(x_i^{N-j})}, \tag{4.1}$$

where N is the number of variables. Given a density distribution $w(x)$ for the eigenvalues of the model one has, in the biorthogonal case (and when $|\lambda|\leq N$)

$$\begin{aligned} <\mathfrak{s}_\lambda>_{w,N,\theta} &= \int\prod_i \mathrm{d}w(x_i)\det\left(x_i^{\lambda_j+N-j}\right)\det\left(x_i^{\theta(N-j)}\right) \\ &= C(N)\det\left\langle x^{\lambda_i+N-i+\theta(N-j)}\right\rangle_{w,\theta}, \end{aligned} \tag{4.2}$$

according to the same trick that is used in [10] and which appears in Eq. (1.1) in [23]. Now, the idea is to use an identity due to Macdonald (see [12], Example I.3.21) which states:

Let $\{h_{rs}\}$ be any collection of commuting indeterminates such that

$$h_{0s} = 1, \quad h_{-1,s} = h{-2,s} = \ldots = 0, \quad \forall s\in\mathbb{Z}_+, \tag{4.3}$$

and set

$$\tilde{\mathfrak{s}}_\lambda = \det \left(h_{\lambda_i - i + j, j-1} \right)_{i,j=1}^{k} \tag{4.4}$$

for $k \geq |\lambda|$. Then one has

$$\tilde{\mathfrak{s}}_\lambda = \det \left(\tilde{\mathfrak{s}}_{(p_i|q_j)} \right), \tag{4.5}$$

where $\lambda = (p_1, \ldots, p_d | q_1, \ldots, q_d)$ in Frobenius notation.

Exactly as in [10], consider the matrix $g = (g_{kl})$ of format $\infty \times N$ with entries $g_{kl} = \left\langle x^{k+\theta l} \right\rangle_{w,\theta}$ with $k \in \mathbb{N}$ and $l = 0, \ldots, N-1$. Because the upper $N \times N$ block (that is, the first N rows) is non-degenerate, one can multiply on the right by its inverse (this is an N-dependent, $\lambda-$independent quantity), and get a lower triangular matrix $g' = (g'_{kl})$ satisfying

$$g'_{kl} = \delta_{kl}, \quad 0 \leq k \leq l \leq N-1, \tag{4.6}$$

and therefore one has

$$< \mathfrak{s}_\lambda >_{w,N,\theta} = B(N) \det(g'_{\lambda_i}). \tag{4.7}$$

Now one can define

$$h_{rs} \equiv \begin{cases} g'_{r-s+N-1, N-s-1}, & s = 0, 1, \ldots, N-1, r \geq 0 \\ \delta_{r0}, & s \geq N, r \geq 0 \\ 0, & s \geq 0, r < 0, \end{cases} . \tag{4.8}$$

Therefore, the biorthogonalization preserves the Giambelli compatibility condition.

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Universitat Politécnica de Catalunya, Department de Física i Enginyeria Nuclear, Comte Urgell 187, E-08036 Barcelona, Spain

E-mail address: `csdt.rmt@gmail.com, tierz@brandeis.edu`