Quantum topology and strongly-correlated systems: the Chern-Simons model

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  - Brief introduction to Chern-Simons (CS) theory
  - Random matrix description of CS theory
  - Sample computations: Partition function and quantum dimensions
- STRONGLY CORRELATED MODELS WITH TOPOLOGICAL PROPERTIES
- CONCLUSIONS AND OUTLOOK

- Let $H = (H_{jk})_{j,k=1}^{N}$ be a square $N \times N$ matrix with randomly distributed elements H_{jk}. This is a random matrix with respect to a probability distribution, defined by:

$$P_{\beta}^{(N)}(H) \propto \exp\left(-\beta \mathrm{Tr} \mathrm{V}(H)\right),$$

- The first and most studied ensembles are the Gaussian ensembles, $V(H) = H^2$. It can be actually shown that the previous expression is automatically restricted to the form

$$P(H) = \exp\left(-a \mathrm{Tr} H^2 + b \mathrm{Tr} H + c\right), \qquad a > 0,$$

if one postulates statistical independence of the matrix elements H_{ij}. There are three different ensembles defined depending on the values of the parameter $\beta = 1, 2$ or 4.

Ensembles of random $N \times N$ matrices H are defined by the following demands:

1. The probability $P(H)d[H]$ is invariant under any transformation $H \to U^{-1}HU$, where U is either an orthogonal ($\beta = 1$), unitary ($\beta = 2$) or symplectic ($\beta = 4$) matrix. That is to say, if $H' = U^{-1}HU$ where U belongs to the unitary group $U(N;\beta)$, then $P(H')d[H'] = P(H)d[H]$.

2. The matrix elements which are not related by the symmetry of the matrix are statistically independent (Gaussian ensembles)

- Diagonalization: for each matrix H there is a matrix U that maps it onto its eigenvalues. The Jacobian of the transformation is $J_\beta(\{x_i\}) = \prod_{i<j} |x_i - x_j|^\beta$. The resulting expression with a generic potential is

$$P(x_1, ..., x_N) = \mathrm{C_N} \prod_{i<j} |x_i - x_j|^\beta \prod_{i=1}^{N} \mathrm{e}^{-\frac{\beta}{2} V(x_i)}.$$

  The potential $V(x) = \log^2 x$ (log-normal weight function $\omega(x) = \mathrm{e}^{-\log^2 x}$) is at the center of most developments in this talk.
- The main relevant quantities are m-partial integrations over the previous N-dimensional probability density function

- A central and powerful result in random matrix theory is that m-point correlation function can be computed from the two-point kernel as follows (simplest case of an Hermitian ($\beta = 2$) ensemble)

$$R_m^{(N)}(x_1, \ldots, x_m) = \det\left(K_N\left(x_i, x_j\right)\right)_{1 \leq i,j \leq m}$$

- Orthogonal polynomials method $\Longrightarrow$ explicit expressions for $K_N\left(x_i, x_j\right)$. Let $p_N(x) = c_N x^N + \ldots$ the Nth orthogonal polynomial associated to $\mathrm{e}^{-V(x)}$, the two-point kernel is

$$\begin{aligned} K_N\left(x, y\right) &= \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \sum_{i=0}^{N-1} p_i\left(x\right) p_i\left(y\right) \\ &= \frac{c_{N-1}}{c_N} \frac{p_N(x) p_{N-1}(y) - p_{N-1}(x) p_N(y)}{x - y} \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \end{aligned}$$

- We consider Chern-Simons theory on a three-manifold M and for a simply-laced gauge group G, with action

$$S(A) = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

where A is a connection on M.

- Witten showed in 1989, that the partition function of Chern-Simons theory

$$Z_k(M) = \int \mathcal{D}A e^{iS_{\mathrm{CS}}(A)},$$

defines a topological invariant .

- The contribution of reducible flat connections to the Chern-Simons partition function of $X(\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n})$ for $U(N)$ is given by (M. Mariño, Comm. Math. Phys. 253, 25 (2004))

$$Z_{\mathrm{CS}}(M) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \mathrm{d}y_i \, \mathrm{e}^{-y_i^2/2g_s - lt_i y_i} \frac{\prod_{j=1}^{n} \prod_{k<l} 2 \sinh \frac{y_k - y_l}{2p_j}}{\prod_{k<l} \left(2 \sinh \frac{y_k - y_l}{2}\right)^{n-2}}$$

- The simplest case is S^3 and gauge group $U(N)$

$$Z_{\mathrm{CS}}(S^3) = \left(\frac{g_s}{2\pi}\right)^{-\frac{N}{2}} \int_{-\infty}^{\infty} \prod_i \frac{du_i}{2\pi} e^{-\frac{u_i^2}{2g_s}} \prod_{i<j} \left(2 \sinh\left(\frac{u_i - u_j}{2}\right)\right)^2$$

- Thus, Mariño ended up with N-dimensional integral expressions for Chern-Simons partition functions whose expression resemble that of random matrix theory.

Chern-Simons theory and the Stieltjes-Wigert matrix model

Partition function three-sphere U(N) (M.T., Mod.Phys.Lett. A19, 1365 (2004))

- Ingredients: 1) Change of variables $e^{u_i} = x_i$ 2) Symmetry of the log-normal $\omega(xq) = \sqrt{q}x\omega(x)$ (when $\omega(x) = e^{-\log^2 x_i/2g_s}$), then

$$Z\left(S^3\right) = \int \prod_{i=1}^{N} \frac{du_i}{2\pi} e^{-\frac{u_i^2}{2g_s}} \prod_{i<j} \left(2\sinh\left(\frac{u_i - u_j}{2}\right)\right)^2$$

$$= (2\pi)^{-N} e^{\frac{-N^3 g_s}{2}} \int \prod_{i=1}^{N} dx_i e^{-\frac{\log^2(x_i)}{2g_s}} \prod_{i<j} (x_i - x_j)^2 .$$

- Last expression is the Stieltjes-Wigert matrix model. For the partition function computation, we actually only need the leading coefficients, $p_i(x) = a_i x^i + \ldots$ which are

$$a_j = q^{(j+1/2)^2} \left\{(1-q)\ldots\left(1-q^j\right)\right\}^{-1/2} .$$

- The partition function in terms of the orthogonal polynomials is:

$$
\begin{aligned}
Z &= \int \ldots \int \prod_{i=1}^{N} \omega\left(x_i\right) dx_i \prod_{i<l} \left(x_i - x_l\right)^2 \\
&= \frac{N!}{\prod_{i=0}^{N-1} a_i^2} = N! a_0^{-2N} \prod_{i=1}^{N-1} \left(\left(\frac{a_{i-1}}{a_i} \right)^2 \right)^{N-i} .
\end{aligned}
$$

- Using the coefficients, we have $\left(\frac{a_{j-1}}{a_j}\right)^2 = q^{-4j}\left(1-q^j\right)$ and $a_0 = q^{1/4}$, leads to

$$
Z_{\mathrm{SW}} = N! q^{-\frac{1}{6}N(2N-1)(2N+1)} \prod_{j=1}^{N-1} \left(1 - q^j\right)^{N-j}
$$

- Then, the N-dimensional integral is computed explicitly:

$$Z_{\text{sinh}} = \left(\frac{g_s}{2\pi}\right)^{N/2} N! \mathrm{e}^{\frac{1}{6}g_s N\left(N^2-1\right)} \prod_{j=1}^{N-1} \left(1-q^j\right)^{N-j},$$

and transforming the product term and identifying $g_s = \frac{2\pi i}{k+N}$ (coupling constant with CS parameter) we finally find:

$$Z\left(S^3\right) = \mathrm{e}^{\frac{1}{4}i\pi N^2} \left(k+N\right)^{-N/2} \prod_{j=1}^{N-1} \left(2\sin\frac{\pi j}{k+N}\right)^{N-j}.$$

- More analytical computations done. The Chern-Simons invariant of the unknot are quantum dimensions. We showed

$$
\begin{aligned}
< \quad \mathfrak{s}_\lambda(M) >_w &= \int [dM] \mathfrak{s}_\lambda(M) \mathrm{e}^{-\frac{1}{2g_s}\mathrm{Tr}(\log M)^2} \\
&= q^{-n|\lambda| - \frac{1}{2} C_\lambda^{U(n)}} \mathcal{D}_\lambda .
\end{aligned}
$$

where $C_\lambda^{U(n)}$ is the Casimir of $U(N)$ and the last term are the quantum dimensions:

$$
\mathcal{D}_\lambda \equiv \prod_{x \in \lambda} \frac{\lfloor n + c(x) \rfloor}{\lfloor h(x) \rfloor},
$$

where for each box of the diagram $h(x) \equiv \lambda_i + \lambda'_j - i - j + 1$ is the hook-length and $c(x) \equiv j - i$ the content of x.

1D Strongly correlated models

Motivation

- The Wigner-Dyson distribution of eigenvalues $P_N^{\beta}(\{x_i\})$ coincides with the probability distribution of the N-particle coordinates $P_N^{\beta}(\{r_i\})$ of the quantum ground state of a $1D$ Hamiltonian: the Calogero model:

$$\Psi_0(x_1, ..., x_N) = C_N \prod_{i<j} |x_i - x_j|^{\beta/2} \prod_{i=1}^{N} \mathrm{e}^{-\frac{\beta}{4}x_i^2},$$

$$H = -\sum_{i=1}^{N} \frac{d^2}{dx_i^2} + \beta\left(\frac{\beta}{2} - 1\right) \sum_{i<j} \frac{1}{(x_i - x_j)^2} + \frac{\beta^2}{4} \sum_{i=1}^{N} x_i^2.$$

- Is there an analogous result for the CS matrix model ? The literature (e.g. V. I. Inozemtsev and D.V. Mescheryakov, Phys. Lett. A, 106, 101 (1984)) seems to say no, but the answer is affirmative.

- The result is (counterexample by Forrester (1994) to Inozemtsev (1989)):

$$H = -\sum_{i=i}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{1}{g_s^2} \sum_{i=1}^{N} x_i^2 + \frac{m(m-1)}{2L} \sum_{i<j} \frac{1}{\sinh^2\left(\frac{x_i - x_j}{2L}\right)} + \frac{m}{g_s L} \sum_{i<j} (x_i - x_j) \coth\left(\frac{x_i - x_j}{2L}\right),$$

- If $\Psi_0 = \mathrm{e}^{-\mathcal{H}}$ then $\mathcal{H} = \frac{1}{2g_s} \sum_{i=1}^{N} x_i^2 - \sum_{i<j} \ln \sinh \left|\frac{x_i - x_j}{2L}\right|$. The last term is the Coulomb potential between charges in the surface of a $2D$ cylinder

- We compute the pairs H, Ψ_0 for the orthogonal and symplectic groups case.
- We show that the CS exactly solvable model behaves as a system of N free fermions at finite temperature
    - The two-point kernel of the model is $K(x,y) = \frac{a\sin(\pi(x-y))}{\sinh(a\pi(x-y))}$ like free fermions at finite T. There is exponential decay of correlations: finite temperature diminishes correlation between particles.
    - The density matrix of a system of 1D free fermions at finite T and harmonic confinement coincides with a probability density that includes the CS density

- Quantum topological invariants can be extracted from Ψ_0 in two ways:
  - As the norm of $|\Psi_0|^2$ as is typical of Chern-Simons theory $Z(S^3) = \langle \Psi_0 | \Psi_0 \rangle$, but with the N-body wavefunctions instead of wavefunctionals.
  - Specifying a very particular configuration where the fermions are equispaced (see next slide discussion of the Sutherland model)
- The hyperbolic Sutherland and the CS Hamiltonians are related by

$$H_{CS} = H_{Suth} + \sum_{i<j} (x_i - x_j) \coth (x_i - x_j)$$

- At the level of the wavefunction they differ by a Gaussian factor. Since this factor in the CS wavefunction contributes a phase, the Sutherland model also leads to the CS partition function when evaluated in a crystalline configuration.

Strongly correlated models: 1D Solvable model

Sutherland model (Preprint)

- Consider the ground state wavefunction of Sutherland model

$$\Psi_0\left(x_1,\ldots,x_N\right)=\prod_{i<j}\left|\sin\frac{\pi\left(x_i-x_j\right)}{L}\right|^{\lambda}$$

- We evaluate the squared-ground state wavefunction in the fixed and equally-spaced configuration $x_i = c - i$. Then, we find that

$$\begin{aligned}\left|\Psi_0\left(c,\ldots,c-N\right)\right|^2_{\lambda=1/2} &= \prod_{i<j}\sin\frac{\pi\left(i-j\right)}{L}\\ &= \prod_{k=1}^{N-1}\left[\sin\frac{\pi}{L}\cdot\sin\frac{2\pi}{L}\cdot\ldots\cdot\sin\frac{\pi\left(N-k\right)}{L}\right]\\ &= \prod_{k=1}^{N-1}\sin^{N-k}\left(\frac{\pi}{L}k\right).\end{aligned}$$

This is the CS partition function as soon as we identify the arbitrary parameter $L = k + N$.

Strongly correlated models: Contact with condensed matter physics

Sutherland model and Luttinger liquid (Preprint)

- Slightly improving a paper by M.A. Cazalilla (J. of Phys. B: 37,S1 (2004)) we show that the low-energy wavefunction of a Luttinger liquid of particles on a box (open boundary conditions) is giving by a C_N-Sutherland model

$$\Phi(x_1, \ldots, x_N) = \prod_{i<j} \left| \sin\left[\frac{\pi(x_i + x_j)}{2L}\right] \sin\left[\frac{\pi(x_i - x_j)}{2L}\right] \right|^{\frac{1}{K}} \times \prod_{i=1}^{N} \sin\left(\frac{\pi x_i}{L}\right)$$

- As in the previous slide, evaluation of the wavefunction in its classical configuration leads now to Chern-Simons partition function on S^3 and $Sp(2N)$.

Strongly correlated models: Contact with condensed matter physics

Laughlin wavefunction on a cylinder (Preprint II)

- Laughlin this geometry is (Thouless 84, Rezayi & Haldane 94). Complex coordinate: $z_j = x_j + iy_j$

$$\Psi_0\left(x_1, \ldots, x_N\right) = \prod_{i=1}^{N} \mathrm{e}^{-\frac{x_i^2}{2g_s}} \prod_{i<j} \left| \sinh \frac{\pi\left(z_i - z_j\right)}{L} \right|^{\lambda}$$

- Chern-Simons model is the 1D version of this Laughlin wavefunction
  - Taking the thin cylinder limit
  - When the magnetic field $B \to \infty$ and one projects to lowest Landau level. Half of the phase space degrees of freedom are frozen. There are two ways of doing this, one leads to the Sutherland model while the other leads to the Chern-Simons model. These two are related by a Fourier transform.

Strongly correlated models: Contact with condensed matter physics

Further approximation: a charged Bose gas (Preprint II)

- If one considers the 1D model and the approximation, $L \longrightarrow 0$, $\sinh\left(\left(x_i - x_j\right)/2L\right) \simeq \exp\left(\left|x_i - x_j\right|/2L\right)$ ($q \longrightarrow 0$ limit of the Chern-Simons model) leads to

$$H = -\sum_{i=i}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \delta\left(x_i - x_j\right) + \sum_{i<j} \left|x_i - x_j\right|,$$

$$\Psi_0\left(x_1, \ldots, x_N\right) = \prod_{i=1}^{N} \mathrm{e}^{-\frac{x_i^2}{2g_s}} \prod_{i<j} \exp\left|x_i - x_j\right|,$$

which is a charged Bose model (Lieb-Liniger model with 1D Coulomb potential $V = \left|x_i - x_j\right|$).

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- The appearance of random matrices in Chern-Simons theory indicates connections with condensed matter models. In particular:
  - Interpretation of the models as a model of Calogero-Suherland type models with different and additional two-body interactions is possible
  - The models are many-body systems with a Coulomb gas interpretation: one-component plasmas in 1D but interacting in the surface of a cylinder. These models have quantum topological properties, inherited from Chern-Simons theory. They are related to Laughlin wavefunctions on the cylinder and to a charged Bose gas model in the $q \longrightarrow 0$ limit.