Chern-Simons matrix models and Stieltjes-Wigert polynomials

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(Received 10 October 2006; accepted 8 December 2006; published online 15 February 2007)

Employing the random matrix formulation of Chern-Simons theory on Seifert manifolds, we show how the Stieltjes-Wigert orthogonal polynomials are useful in exact computations in Chern-Simons matrix models. We construct a biorthogonal extension of the Stieltjes-Wigert polynomials, not available in the literature, necessary to study Chern-Simons matrix models when the geometry is a lens space. We also study the relationship between Stieltjes-Wigert and Rogers-Szegö polynomials and the corresponding equivalence with a unitary matrix model. Finally, we give a detailed proof of a result that relates quantum dimensions with averages of Schur polynomials in the Stieltjes-Wigert ensemble. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2436734]

I. INTRODUCTION

In the late 1980s,[1] Witten considered a topological gauge theory for a connection on an arbitrary three-manifold M, based on the Chern-Simons (CS) action,

$$S_{\mathrm{CS}}(A)=\frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right), \tag{1.1}$$

with k an integer number. One of the most important aspects of Chern-Simons theory is that it provides a physical approach to three dimensional topology. In particular, it gives three-manifold invariants and knot invariants. For example, the partition function,

$$Z_k(M)=\int \mathcal{D}A e^{iS_{\mathrm{CS}}(A)}, \tag{1.2}$$

delivers a topological invariant of M, the so-called Reshetikhin-Turaev-Witten invariant. Recent reviews are Refs. 2 and 3.

As reviewed in Ref. 3, a great deal of interest has been focused on the fact that Chern-Simons theory provides large N duals of topological strings. This connection between Chern-Simons theory and topological strings was already pointed out by Witten[5] (see also Ref. 6), and then extended in Ref. 7.

Recent progress in Chern-Simons theory includes a description of Chern-Simons theory on certain geometries in terms of models of random matrices. Consider the partition function of Chern-Simons theory on a Seifert space $M=X(p_1/q_1,\ldots,p_n/q_n)$. This is obtained by doing sur-

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gery on a link in S^3 with $n+1$ components, out of which n are parallel, unlinked unknots, and one has link number 1 with each of the n unknots. The surgery data are p_j/q_j for the unlinked unknots, $j=1,\ldots,n$, and 0 for the last component. The partition function is[4] (see Appendix C for details on the notation)

$$\mathbf{Z}_{\mathrm{CS}}(M)=\frac{(-1)^{|\Delta_+|}}{|\mathcal{W}|(2\pi i)^r}\left(\frac{\mathrm{Vol}\Lambda_{\mathrm{w}}}{\mathrm{Vol}\Lambda_{\mathrm{r}}}\right)\frac{[\mathrm{sgn}(P)]^{|\Delta_+|}}{|P|^{r/2}}\mathrm{e}^{(\pi id/4)\mathrm{sgn}(H/P)-(\pi idy/12l)\phi}$$

$$\times\sum_{t\in\Lambda_{\mathrm{r}}/H\Lambda_{\mathrm{r}}}\int d\beta \mathrm{e}^{-\beta^2/2g_s-lt\beta}\frac{\prod_{i=1}^{n}\prod_{\alpha>0}2\sinh(\beta\alpha/2p_i)}{\prod_{\alpha>0}(2\sinh(\beta\alpha/2))^{n-2}}. \tag{1.3}$$

This expression gives the contribution of the reducible flat connections to the partition functions. Recall that for both S^3 and lens spaces this amounts to the exact partition function. The case $n=0$ corresponds to the three-sphere S^3 that leads to Eq. (1.5). Thus, for the case of $U(N)$, and focusing on a particular sector of flat connections, we get the following matrix model:

$$Z_{\mathrm{CS}}(M)=\prod_{i=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}y_i\mathrm{e}^{-y_i^2/2g_s-lt_iy_i}\frac{\prod_{j=1}^{n}\prod_{k<l}2\sinh[(y_k-y_l)/2p_j]}{\prod_{k<l}(2\sinh[(y_k-y_l)/2])^{n-2}}. \tag{1.4}$$

Of course, the simplest case is that of S^3 with gauge group $U(N)$, which is given by the partition function of the following random matrix model:

$$Z=\frac{\mathrm{e}^{-(g_s/12)N(N^2-1)}}{N!}\int\prod_{i=1}^{N}\mathrm{e}^{-u_i^2/2g_s}\prod_{i<j}\left(2\sinh\frac{u_i-u_j}{2}\right)^2\frac{du_i}{2\pi}. \tag{1.5}$$

From the point of view of topological strings, this describes open topological A strings on T^*S^3 with N branes wrapping S^3.[4] This latter case, as shown in Ref. 8, can be studied with usual techniques of random matrix theory. More precisely, the Stieltjes-Wigert polynomials, a member of the q-deformed orthogonal polynomial family,[9] allow us to compute, in exact fashion, quantities associated with the matrix model. In the computation, the q-parameter of the polynomials turns out to be naturally identified with the q-parameter of the quantum group invariants associated with the Chern-Simons theory. This is so because the previous model can be easily mapped into

$$Z=\int[dM]\mathrm{e}^{-(1/2g_s)\mathrm{Tr}(\log M)^2}, \tag{1.6}$$

named Stieltjes-Wigert ensembles, after the associated orthogonal polynomials.

Chern-Simons matrix models have been further considered in Refs. 10–17 and also play a central role in q-2D (two-dimensional) Yang-Mills theory.[18–23] Most of these works focus on the relevance to topological strings. In Refs. 8 and 10, the emphasis is on exact solutions and on the special features of the matrix models. The works of Caporaso *et al.*[19,20,23] also make an extensive use of the properties of the Stieltjes-Wigert orthogonal polynomials. We shall be focusing here on the aspects of the Chern-Simons matrix models that have to do with the associated system of orthogonal polynomials.

This paper is organized as follows. In the next section we shall construct a biorthogonal extension of the Stieltjes-Wigert polynomials, in order to study the matrix model [Eq. (1.4)] when $n=1$ and $n=2$. These polynomials have not been discussed in the (vast) orthogonal polynomials literature, so most of our effort is on their derivation and to establish some of its fundamental properties. They are necessary if one wants to obtain full analytic results when the geometry is something more complicated than S^3. Note that matrix models in the lens space case have already been studied (with loop equations),[13] but if one desires an all order result as in Ref. 8, the knowledge of orthogonal polynomials is then necessary. After the construction of the biorthogonal Stieltjes-Wigert polynomials in Sec. II, we discuss some of their mathematical properties in Sec. II A. In Secs. III and IV, we discuss several aspects of the Chern-Simons matrix models by focusing on the properties of the (ordinary) Stieltjes-Wigert polynomials. In particular, in Sec. III we employ the intimate relationship between Stieltjes-Wigert and Rogers-Szegö polynomials to find the exact relation between the Stieltjes-Wigert matrix model and Okuda's unitary matrix model[15] (see Sec. III for its definition). Finally, in the last section we give a detailed proof, employing a mixture of combinatorial and orthogonal polynomial results, of the equality between quantum dimensions and averages of Schur polynomials in the Stieltjes-Wigert ensemble.[2] We conclude with a summary and with some avenues for further research, presented in the Conclusions and Outlook.

II. BIORTHOGONAL STIELTJES-WIGERT

Let us consider the generic expression [Eq. (1.4)] in the $n=1$ and $n=2$ cases, which correspond to the case of lens spaces. We are led to a biorthogonal extension of the S^3 model,

$$Z=\int\prod_{i=1}^{N}\mathrm{e}^{-u_i^2/2g_s}\prod_{i<j}\left(2\sinh\frac{u_i-u_j}{2P}\right)\left(2\sinh\frac{u_i-u_j}{2Q}\right)\frac{du_i}{2\pi}. \tag{2.1}$$

Recall that a biorthogonal ensemble of random matrices has the probability density [24]

$$P(x_1,\ldots,x_N)=C_N\prod_{i=1}^{N}\omega(x_i)\prod_{i<j}(x_i-x_j)(x_i^k-x_j^k), \tag{2.2}$$

where k is a fixed real number. In total analogy with the usual Hermitian case $(k=1)$ one can study Eq. (2.2) by considering a pair of biorthogonal polynomials,

$$\int\omega(x)Y_n(x,k)Z_m(x,k)dx=h_{n,k}\delta_{n,m}, \tag{2.3}$$

with

$$\begin{aligned}\int Y_n(x,k)x^{kj}\omega(x)dx&=\alpha_n^{(k)}\delta_{n,j},\\ \int Z_n(x,k)x^{j}\omega(x)dx&=\beta_n^{(k)}\delta_{n,j}.\end{aligned} \tag{2.4}$$

We warn the reader that the term *biorthogonal* is employed in different contexts in the literature. The classical cases (Hermite, Laguerre, and Jacobi) were worked out in Ref. 24. Note that Eq. (2.2) is exactly the type of ensemble that Eq. (2.1) leads us to consider since

$$\begin{aligned}Z^{P,Q}&=\int\prod_i\frac{\mathrm{d}u_i}{2\pi}\mathrm{e}^{-u_i^2/2g_s}\prod_{i<j}\left(2\sinh\left(\frac{u_i-u_j}{2P}\right)\right)\left(2\sinh\left(\frac{u_i-u_j}{2Q}\right)\right)\\ &=q^{-N\alpha^2/2}\int\prod_i\frac{\mathrm{d}y_i}{2\pi}\mathrm{e}^{-\kappa^2\log^2 y_i}\prod_{i<j}(y_i^{1/P}-y_j^{1/P})(y_i^{1/Q}-y_j^{1/Q}),\end{aligned} \tag{2.5}$$

with $u_i=\log \mathrm{e}^{\alpha/2\kappa^2}y_i, \kappa^2=1/2g_s$, and $\alpha=-1-\beta(N-1)/2, \beta=1/P+1/Q$. Finally, with $y_i=\mathrm{e}^{(P-1)/2\kappa^2P}x_i^P$ and some rewriting,

$$Z^{P,Q} = P^N e^{-N/4\kappa^2(1/P + \beta(N-1)/2)^2} \int \prod_i \frac{dw_i}{2\pi} e^{-\kappa^2 P^2 \log^2 x_i} \prod_{i<j} (x_i - x_j)(x_i^{P/Q} - x_j^{P/Q}), \tag{2.6}$$

which is of the form Eq. (2.2) with the log-normal (Stieltjes-Wigert) weight function: $\omega(x) = e^{-\kappa^2 P^2 \log^2 x_i}$, the q-parameter is then $q = e^{-1/2\kappa^2 P^2} = e^{-g_s/P^2}$. Therefore, if we want to go beyond the S^3 case one has to construct the biorthogonal Stieltjes-Wigert polynomials, not available in the literature. Thus, this is our main task in what follows. The method we have chosen is based on a simple but fundamental result by Askey, which relates the q-Laguerre orthogonal polynomials and the Stieltjes-Wigert polynomials,[25]

$$\lim_{\alpha\to\infty} L_n^{\alpha}(q^{-\alpha}x;q) = S_n(x;q), \tag{2.7}$$

and then we take into account the biorthogonal construction of the q-Laguerre polynomials, carried out by Al-Salam and Verma in the early 1980's. The resulting polynomials were named q-Konhauser as they could also be interpreted as a q-deformed version of the biorthogonal Laguerre polynomials, worked out by Konhauser. The Stieltjes-Wigert (SW) polynomials are[27] (in Ref. 25 they appear, in Eq. (2.5), slightly reformulated)

$$S_n(x|q) \equiv \frac{1}{(q;q)_n} \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q (-1)^r q^{r^2} x^r. \tag{2.8}$$

The limit (2.7) will provide us with a biorthogonal extension of the SW polynomials starting with the q-Konhauser polynomials. Therefore, following Ref. 26, let us write

$$Z_n^{(\alpha)}(x,k|q) \equiv \frac{[q^{1+\alpha}]_{nk}}{(q^k;q^k)_n} \sum_{j=0}^{n} \frac{(q^{-nk};q^k)_j q^{1/2[kj(kj-1)]+kj(n+\alpha+1)}}{(q^k;q^k)_j [q^{1+\alpha}]_{kj}} x^{kj} \tag{2.9}$$

and

$$Y_n^{(\alpha)}(x,k|q) \equiv \frac{1}{[q]_n} \sum_{r=0}^{n} \frac{x^r q^{(1/2)[r(r-1)]}}{[q]_r} b_r^{\alpha}, \tag{2.10}$$

with

$$b_r^{\alpha} \equiv \sum_{s=0}^{r} \frac{[q^{-r}]_s}{[q]_s} q^s (q^{1+\alpha+s};q^k)_n. \tag{2.11}$$

These polynomials satisfy

$$\langle Z_n^{(\alpha)}(x,k|q), Y_m^{(\alpha)}(x,k|q) \rangle = k_n^{(\alpha)} \delta_{n,m} \quad \text{with } k_n^{(\alpha)} = \frac{[q^{1+\alpha}]_{nk} q^{-nk}}{[q]_n}, \tag{2.12}$$

with respect to the normalized q-Laguerre measure,[26]

$$\omega(x) = \frac{\Gamma_q(\alpha)}{\Gamma(-\alpha)\Gamma(1+\alpha)(1-q)^{\alpha+1}} \frac{x^{\alpha}}{[-x]_{\infty}}, \quad \alpha > -1. \tag{2.13}$$

We have to study

$$Z_n(x,k|q) \equiv \lim_{\alpha\to\infty} Z_n^{(\alpha)}(q^{-\alpha}x,k|q),$$
$$Y_n(x,k|q) \equiv \lim_{\alpha\to\infty} Y_n^{(\alpha)}(q^{-\alpha}x,k|q). \tag{2.14}$$

In the first case, one readily finds

$$Z_n(x,k|q) = \frac{1}{(q^k;q^k)_n}\sum_{j=0}^{n}\frac{(q^{-nk};q^k)_j q^{1/2[kj(kj-1)]+kj(n+1)}}{(q^k;q^k)_j}x^{kj}, \tag{2.15}$$

which can be conveniently reexpressed as

$$Z_n(x,k|q) = \frac{1}{(q^k;q^k)_n}\sum_{r=0}^{n}\begin{bmatrix} n \\ r\end{bmatrix}_{q^k}(-1)^r q^{1/2[r^2k(k+1)]}x^{kr}. \tag{2.16}$$

Regarding $Y_n(x,k|q)$ we have to find $b_r \equiv \lim_{\alpha\to\infty} q^{-\alpha r}b_r^{\alpha}$. Employing q-Taylor[26] one can write

$$(q^{1+\alpha}x;q^k)_n = \sum_{r=0}^{n}\frac{x^r[1/x]_r}{[q]_r}\sum_{s=0}^{r}\frac{[q^{-r}]_s}{[q]_s}q^s(q^{1+\alpha+s};q^k)_n; \tag{2.17}$$

therefore

$$(qx;q^k)_n = \sum_{r=0}^{n}\frac{x^r[q^{\alpha}/x]_r}{[q]_r}q^{-\alpha r}b_r^{\alpha}. \tag{2.18}$$

Taking the $\alpha\to\infty$ limit and using the finite q-binomial theorem,[9]

$$(qx;q^k)_n = \sum_{r=0}^{n}\frac{x^r}{[q]_r}b_{n,r} = \sum_{r=0}^{n}(-1)^r\begin{bmatrix} r \\ n\end{bmatrix}_{q^k}q^{1/2[kr(r-1)]+r}x^r, \tag{2.19}$$

so that

$$\frac{b_{n,r}}{[q]_r} = (-1)^r\begin{bmatrix} n \\ r\end{bmatrix}_{q^k}q^{1/2[kr(r-1)]+r}. \tag{2.20}$$

For later use note that one also has

$$\frac{b_{n,r}}{[q]_r} = \frac{1}{r!}\left(\frac{d}{dx}\right)^{(r)}(qx;q^k)_n|_{x=0}. \tag{2.21}$$

From this one gets

$$Y_n(x,k|q) = \frac{1}{[q]_n}\sum_{r=0}^{n}(-1)^r\begin{bmatrix} n \\ r\end{bmatrix}_{q^k}q^{1/2[r(r+1)]+1/2[kr(r-1)]}x^r. \tag{2.22}$$

For $k=1$, both polynomials reduce to the Stieltjes-Wigert polynomials [Eq. (2.8)]. Writing $Y_n(x,k|q)=y_{n,k}x^n+\cdots$ and $Z_n(x,k|q)=z_{n,k}x^{nk}+\cdots$ one finds

$$z_{n,k} = \frac{(q^{-nk};q^k)_n}{(q^k;q^k)_n^2}q^{1/2[kn(kn-1)]+kn(n+1)} = \frac{(-1)^n q^{1/2[n^2k(k+1)]}}{(q^k;q^k)_n} \tag{2.23}$$

and

$$y_{n,k} = \frac{1}{n![q]_n} q^{1/2[n(n-1)]} \left(\frac{d}{dx}\right)^{(n)} (qx;q^k)_n|_{x=0} = \frac{(-1)^n q^{1/2(k+1)n(n-1)+n}}{[q]_n}. \tag{2.24}$$

This leads to

$$\langle Y_n(x,k|q), Z_m(x,k|q)\rangle = h_n \delta_{n,m}, \tag{2.25}$$

with respect to the measure $A dx/[-x]_\infty[-q/x]_\infty$, with A such that $\langle 1,1\rangle = 1$ and

$$h_n = \frac{q^{-nk}}{[q]_n}. \tag{2.26}$$

Using this, we can find, for example,

$$Z^{P,Q} = N!\left(\frac{g_s}{2\pi}\right)^{N/2} q^{-N/2P^2[-(1+1/2(1+P/Q)(N-1))^2+1+4/3(N^2-1)]} \prod_{j=1}^{N-1} (1-q^{j/PQ})^{N-j}, \tag{2.27}$$

which reduces to the known formula when $P=Q=1$.[1,8] One can obtain Eq. (2.26) from $k_n^{(\alpha)}$ in Eq. (2.12) by taking the limit $\alpha\to\infty$. However, it is interesting to obtain the corresponding biorthogonal polynomials as well, whose properties we study in the next section.

A. Mathematical properties of the biorthogonal polynomials

Since the biorthogonal Stieltjes-Wigert polynomials have not been addressed in the literature, we derive here some of its fundamental properties.

1. Behavior under dilatation

First, we find some generating functions for $Z_n(x;k|q)$ and $Y_n(x;k|q)$ $(t\neq q^{-k})$,

$$\sum_{n\geqslant 0} Z_n(x;k|q)t^n = \frac{f(tx^k)}{(t;q^k)_\infty} \tag{2.28}$$

and

$$\sum_{n\geqslant 0} \frac{[q]_n}{(q^k;q^k)_n} Y_n(x;k|q)t^n = \frac{g(tx)}{(t;q^k)_\infty}, \tag{2.29}$$

with

$$f(z) = \sum_{r\geqslant 0} \frac{q^{1/2[r^2k(k+1)]}}{(q^k;q^k)_r}(-z)^r \quad \text{and} \quad g(z) = \sum_{r\geqslant 0} \frac{q^{r(r-1)/2}}{r!} z^r \left(\frac{d}{dx}\right)^{(r)} (qx;q^k)_n|_{x=0}. \tag{2.30}$$

We rely for this on formula (4.2) from Ref. 26. The expression for Z is essentially property (4.1) in Ref. 26. For Eq. (2.29) we use the explicit expression obtained for $Y_n(x;k|q)$. Let us introduce the moment generating function,

$$g(t,x) \equiv \sum_{n\geqslant 0} \frac{[q]_n}{(q^k;q^k)_n} Y_n(x;k|q)t^n, \tag{2.31}$$

which can be written as

$$g(t,x) = \sum_{r\geqslant 0} \frac{q^{r(r-1)/2}}{r!} x^r \left(\frac{d}{dx}\right)^{(r)} \sum_{n\geqslant r} \left.\frac{(qx;q^k)_n t^n}{(q^k;q^k)_n}\right|_{x=0} = \sum_{r\geqslant 0} \frac{q^{r(r-1)/2}}{r!} x^r \left(\frac{d}{dx}\right)^{(r)} \sum_{n\geqslant 0} \left.\frac{(qx;q^k)_n t^n}{(q^k;q^k)_n}\right|_{x=0}$$
$$= \sum_{r\geqslant 0} \frac{q^{r(r-1)/2}}{r!} x^r \left(\frac{d}{dx}\right)^{(r)} \frac{(qxt;q^k)_\infty}{(t;q^k)_\infty} = \frac{1}{(t;q^k)_\infty} \sum_{r\geqslant 0} \frac{q^{r(r-1)/2}}{r!} (xt)^r \left(\frac{d}{dx}\right)^{(r)} (qx;q^k)_n\big|_{x=0}, \quad t \neq q^{-k}. \tag{2.32}$$

In the second line, the extra piece we add, being a degree $r-1$ polynomial in x does not contribute due to the derivative. In the third line we use the q-binomial theorem, and in the fourth one we make the change of variable $x\rightarrow xt$.

Now, by taking Eq. (2.29) with $x\rightarrow \lambda x$, introducing in the right-hand side (rhs) the factor $(\lambda t;q^k)_\infty/(t;q^k)_\infty$ and matching the coefficients of t^n on both sides one gets

$$Y_n(\lambda x;k|q) = \sum_{j=0}^{n} \gamma_{nj}(\lambda) Y_j(x;k|q), \tag{2.33}$$

with

$$\gamma_{nj}(\lambda) = \frac{[q]_j}{[q]_n} \frac{(q^k;q^k)_n}{(q^k;q^k)_j} \frac{\lambda^j(\lambda;q^k)_{n-j}}{(q^k;q^k)_{n-j}}. \tag{2.34}$$

For $k=1$ this reduces to

$$\gamma_{nj}(\lambda) = \lambda^j \frac{[\lambda]_{n-j}}{[q]_{n-j}}, \tag{2.35}$$

and, in particular, one can define

$$\gamma_{nj}(q^{-1}) = q^{-n}(\delta_{n,j} - \delta_{n-1,j}). \tag{2.36}$$

Similar steps involving Eq. (2.28) give

$$Z_n(\lambda x;k|q) = \sum_{j=0}^{n} \zeta_{nj}(\lambda) Z_j(x;k|q), \tag{2.37}$$

with

$$\zeta_{nj}(\lambda) = \frac{1}{(q^k;q^k)_{n-j}} \lambda^{kj}(\lambda^k;q^k)_{n-j}. \tag{2.38}$$

One has $\zeta_{nj}=\gamma_{nj}$ for $k=1$ as it should. Moreover $\gamma_{nn}=\zeta_{nn}=\lambda^n$, by matching the dominant coefficients on both sides. Note that Eq. (4.2) in Ref. 26 contains a typo as it does not fulfill this last condition (it would give $\zeta_{nn}=1$). And of course one has $\zeta_{nj}(1)=\gamma_{nj}(1)=\delta_{nj}$.

Even though ζ_{nj} and γ_{nj} are defined for $j\leqslant n$ we extend for convenience their definition through

$$\zeta_{nj} = \gamma_{nj} = 0 \quad \text{if } j > n. \tag{2.39}$$

2. Recurrence formulas

The Stieltjes-Wigert polynomials [Eq. (2.8)] satisfy

$$S_{n-1}(x|q) = (1-q^n)S_n(x|q) + xq^n S_{n-1}(xq|q), \tag{2.40}$$

an identity used by Chihara in Ref. 28, to prove that the zeros of the polynomials satisfy

$$x_{n,m} < x_{n-1,m} < q x_{n,m+1}, \tag{2.41}$$

where n denotes the order of the polynomial and m indexes the zero. Note that the zeros of the SW polynomials are an interesting quantity in the context of topological strings.[29] In what follows, we find the same identities for the biorthogonal polynomials.

Fundamental recurrence relation. Note that for the particular value $\lambda = q^{-1}$ Eq. (2.38) gives

$$\zeta_{nj}(q^{-1}) = q^{-kn}(\delta_{n,j} - \delta_{n-1,j}). \tag{2.42}$$

This implies the following simple recurrence relation for $Z_n(x,k|q)$:

$$Z_n(x,k|q) - Z_{n-1}(x,k|q) = q^{kn} Z_n(q^{-1}x,k|q). \tag{2.43}$$

Certainly, if one writes the $Z_n(x,k|q) = \sum_{j=0}^{n} \tau_{n,j} x^{kj}$, it can be checked directly, from the explicit expression in Eq. (2.16), that one has

$$\tau_{n,j} - \tau_{n-1,j} = q^{k(n-j)} \tau_{n,j}, \tag{2.44}$$

which implies the recurrence relation

For $k=1$, Eq. (2.43) reduces to the following relation for the Stieltjes-Wigert polynomials:

$$S_n(y) - S_{n-1}(y) = q^n S_n(q^{-1}y). \tag{2.45}$$

Moment generating recurrences. From Eq. (2.20) one has ($b_{n,0}=1$, $b_{n,-1} \equiv 0$)

$$\frac{b_{n+1,r}}{[q]_r} = \frac{b_{n,r}}{[q]_r} - q^{nk+1} \frac{b_{n,r-1}}{[q]_{r-1}}, \tag{2.46}$$

which implies the following recurrence relation for the $Y_n(x,k|q)$.

$$(1-q^{n+1}) Y_{n+1}(x,k|q) = Y_n(x,k|q) - q^{nk+1} x Y_n(qx,k|q), \tag{2.47}$$

or, equivalently,

$$x Y_n(x,k|q) = q^{-nk}(Y_n(q^{-1}x,k|q) - (1-q^{n+1}) Y_{n+1}(q^{-1}x,k|q)). \tag{2.48}$$

We proceed in analogous way for $Z_n(x;k|q)$. For convenience we introduce coefficients $c_{n,r}$, such that [one has $\tau_{n,r} = (q^{(1/2)[kr(kr-1)]}/(q^k;q^k)_n)/(c_{n,r}/(q^k;q^k)_r)$]

$$Z_n(x,k|q) \equiv \frac{1}{(q^k;q^k)_n} \sum_{r=0}^{n} \frac{x^{kr} q^{1/2[kr(kr-1)]}}{(q^k;q^k)_r} c_{n,r}, \tag{2.49}$$

that is,

$$\frac{c_{n,r}}{(q^k;q^k)_r} = \frac{(q^{-nk};q^k)_r q^{kr(n+1)}}{(q^k;q^k)_r}. \tag{2.50}$$

Then, as in the previous case $c_{n,0}=1$, $c_{n,-1} \equiv 0$, then

$$\frac{c_{n+1,r}}{(q^k;q^k)_r} = \frac{c_{n,r}}{(q^k;q^k)_r} - q^{nk+k} \frac{c_{n,r-1}}{(q^k;q^k)_{r-1}}, \tag{2.51}$$

and one gets the recurrence relation for $Z_n(x;k|q)$,

$$(1-q^{k(n+1)}) Z_{n+1}(x,k|q) = Z_n(x,k|q) - q^{nk+[k(k+1)/2]} x^k Z_n(q^k x,k|q), \tag{2.52}$$

equivalently,

$$x^k Z_n(x,k|q) = q^{-nk+[k(k-1)/2]}(Z_n(q^{-k}x,k|q) - (1-q^{k(n+1)})Z_{n+1}(q^{-k}x,k|q)). \tag{2.53}$$

One can easily check that these recurrence relations both reduce, taking $k=1$, to Eq. (2.40).

To conclude this section, since we know the explicit behavior of the polynomials under dilatation, we can employ Eqs. (2.34) and (2.38), and then Eqs. (2.48) and (2.53), to obtain an explicit way to compute the moments $\langle x^l Y_n(x,k|q) Z_m(x,k|q)\rangle$. For instance, one has

$$\begin{aligned}\langle xY_n(x,k|q)Z_m(x,k|q)\rangle &= q^{-nk}k_m(\gamma_{n,m}(q^{-1}) - (1-q^{n+1})\gamma_{n+1,m}(q^{-1}))\\ &= \frac{q^{-(n+m)k}}{[q]_m}(\gamma_{n,m}(q^{-1}) - (1-q^{n+1})\gamma_{n+1,m}(q^{-1})\ .\end{aligned} \tag{2.54}$$

III. FROM STIELTJES-WIGERT TO ROGERS-SZEGÖ: UNITARY MATRIX MODEL

The Stieltjes-Wigert matrix model posseses distinctive mathematical features, in comparison with other, more usual models in the literature, such as matrix models with Gaussian or polynomial potentials. The log-normal weight function leads to an indeterminate moment problem[30,31] and consequently, the Stieltjes-Wigert polynomials are not dense in $L^2(x,w(x))$ (see Refs. 8 and 10 for details). One of the consequences, discussed in Ref. 10, is the discretization of the Chern-Simons matrix model. The result in Ref. 10 was (including here normalization constants)

$$\begin{aligned}&\left(\frac{g_s}{2\pi}\right)^{-N/2}\int_0^{+\infty}\prod_i \frac{du_i}{2\pi}e^{-u_i^2/2g_s}\prod_{i<j}\left(2\sinh\left(\frac{u_i-u_j}{2}\right)\right)^2\\ &= \left(\frac{q^{-1/2(1-2N+3N^2)}}{[-q^{3/2-N}]_\infty[-q^{N-1/2}]_\infty[q]_\infty}\right)^N \sum_{n_1,\ldots,n_N=-\infty}^{+\infty} e^{-(g_s/2)\Sigma_i n_i^2}\prod_{j<k}\left(2\sinh\left(\frac{g_s}{2}(n_j-n_k)\right)\right)^2.\end{aligned} \tag{3.1}$$

Note the inversion of the coupling constant between the left-hand side (lhs) and rhs.

In addition to this equivalence, we also can find a unitary matrix model that appears in $S^3U\ (N)$ Chern-Simons theory,[15]

$$\widetilde{Z}_{\rm CS} \equiv \int_{U(N)} dU \det \Theta_{00}(U;q) = \frac{1}{|W|}\int\left(\prod_{i=1}^N \frac{d\theta_i}{2\pi}\Theta_{00}(e^{i\theta_i}|q)\right)\prod_{i<j}\sin^2\left(\frac{\theta_i-\theta_j}{2}\right), \tag{3.2}$$

where the integral is taken with respect to the Haar measure over $U(N)$, and the theta function is

$$\Theta_{00}(e^{i\theta}|q) = \sum_{j\in\mathbb{Z}} q^{(j^2/2)}e^{ij\theta}. \tag{3.3}$$

The Stieltjes-Wigert polynomials turn out to be intimately related to the Rogers-Szegö polynomials,[32,33] that are orthogonal on the unit circle. This is useful to establish in detail the exact relationship between the Stieltjes-Wigert matrix model and the unitary model. Following Ref. 33, we recall the definition and relations between the Rogers-Szegő and Stieltjes-Wigert polynomial. The Rogers-Szegő polynomials are defined as

$$H_n(z|q) \equiv \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q z^k, \tag{3.4}$$

and they satisfy an orthogonality relation on the complex unit circle,

$$\frac{1}{2i\pi}\oint_{|w|=1} H_m(-q^{-1/2}\bar{w}|q)H_n(-q^{-1/2}w|q)\Theta_3\left(\frac{\log w}{2i}\Big|\sqrt{q}\right)\frac{dw}{w} = \frac{[q]_m}{q^m}\delta_{mn}, \tag{3.5}$$

where $\Theta_3(z|q)$ is the third Jacobi theta function.[34] Note that the orthogonality coefficients $h_m = [q]_m/q^m$ are identical to the ones (Stieltjes-Wigert) that directly give the Chern-Simons partition function in the $S^3U(N)$ case.[8] This is enough to write down a unitary matrix model for the Chern-Simons partition function. However, let us show this point with detail. The polynomials are also orthogonal with respect to a measure defined on the full real line,[33]

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} H_m(-q^{-1/2}e^{-2i\mu x}|q)H_n(-q^{-1/2}e^{2i\mu x}|q)e^{-x^2}\mathrm{d}x = \frac{[q]_m}{q^m}\delta_{mn}, \tag{3.6}$$

introducing μ through $q \equiv \mathrm{e}^{-2\mu^2}$. Now consider the Stieltjes-Wigert polynomials $S_n(x)$,[27]

$$\begin{aligned} S_n(x) &= \frac{(-1)^n q^{(n/2)+1/4}}{\sqrt{[q]_n}}\sum_{\nu=0}^{n}\begin{bmatrix} n \\ \nu \end{bmatrix}_q q^{\nu^2}(-\sqrt{q}x)^{\nu} \\ &= \frac{(-1)^n q^{(n/2)+1/4}}{\sqrt{[q]_n}}\hat{S}_n(-\sqrt{q}x|q) \quad \text{with } \hat{S}_n(z|q) \equiv \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2}z^k. \end{aligned} \tag{3.7}$$

These polynomials fulfill the following orthogonality relation on the real line:

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \hat{S}_m(-q^{-1/2}e^{-2\mu x}|q)\hat{S}_n(-q^{-1/2}e^{-2\mu x}|q)e^{-x^2}\mathrm{d}x = \frac{[q]_m}{q^m}\delta_{mn}. \tag{3.8}$$

Using an elementary property of the q-binomial coefficients, the two equivalent relationship follow:

$$H_n(x|q^{-1}) = \hat{S}_n(q^{-n}x|q) \text{ and } \hat{S}_n(x|q^{-1}) = H_n(q^{-n}x|q). \tag{3.9}$$

Then

$$\begin{aligned} \langle p_n, p_m\rangle_w &= \rho_{m,n}\frac{k}{\sqrt{\pi}}\int_0^{\infty} \mathrm{e}^{-k^2\log^2 z}\hat{S}_m(-q^{1/2}z)\hat{S}_n(-q^{1/2}z)\mathrm{d}z \\ &= \rho_{m,n}q^{-1/2}\frac{k}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-k^2(x-1/2k^2)^2}\hat{S}_n(-q^{1/2}\mathrm{e}^x)\hat{S}_n(-q^{1/2}\mathrm{e}^x)\mathrm{d}x \\ &= \rho_{m,n}q^{-1/2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\hat{S}_n(-q^{-1/2}\mathrm{e}^{-x/k})\hat{S}_n(-q^{-1/2}\mathrm{e}^{-x/k})\mathrm{d}x \\ &= \rho_{m,n}q^{-1/2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-y^2}H_n(-q^{-1/2}\mathrm{e}^{2i\mu x})H_m(-q^{-1/2}\mathrm{e}^{-2i\mu x})\mathrm{d}y \\ &= \rho_{m,n}q^{-1/2}\frac{1}{2i\pi}\oint_{|w|=1} H_m(-q^{-1/2}\bar{w}|q)H_n(-q^{-1/2}w|q)\Theta_3\left(\frac{\log w}{2i}|\sqrt{q}\right)\frac{\mathrm{d}w}{w} \\ &= \rho_{m,n}q^{-1/2}\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}H_m(-q^{-1/2}\mathrm{e}^{-i\theta}|q)H_n(-q^{-1/2}\mathrm{e}^{i\theta}|q)\Theta_3\left(\frac{\theta}{2}|\sqrt{q}\right), \end{aligned} \tag{3.10}$$

where we also have used, between lines 3 and 4, the fact that $\hat{S}_n(a\mathrm{e}^{-2\kappa x}|q)$ and $H_n(a\mathrm{e}^{2i\kappa y}|q)$ are related by a Fourier transform.[33] We also introduced $2\mu = 1/\kappa$ and

$$\rho_{m,n} = (-1)^{m+n}\frac{q^{(m+n+1)/2}}{\sqrt{[q]_m[q]_n}}. \tag{3.11}$$

The next line is given by the results of the previous section. Now consider Eq. (3.22) in Ref. 15, it reads

$$\widetilde{Z}_{\rm CS} = \frac{1}{|W|}\int\left(\prod_{i=1}^{N}\frac{{\rm d}\theta_i}{2\pi}\Theta_{00}(e^{i\theta_i}|q)\right)\prod_{i<j}\left(\sin\left(\frac{\theta_i-\theta_j}{2}\right)\right)^2$$
$$= \frac{(-1)^{N(N-1)/2}}{|W|}\int\left(\prod_{i=1}^{N}\frac{{\rm d}\theta_i}{2\pi}\Theta_{00}(e^{i\theta_i}|q)\right)\prod_{i<j}(e^{i\theta_i}-e^{i\theta_j})\prod_{i<j}(e^{-i\theta_i}-e^{-i\theta_j})$$
$$= \frac{(-1)^{N(N-1)/2}}{|W|}\int\left(\prod_{i=1}^{N}\frac{{\rm d}\theta_i}{2\pi}\Theta_{00}(e^{i\theta_i}|q)\right)\det_{1\le i,j\le N}(H_{j-1}(e^{i\theta_i}))\det_{1\le i,j\le N}(H_{j-1}(e^{-i\theta_i})). \quad (3.12)$$

Then, considering that Refs. 15 and 33 have different conventions for the third Jacobi function one sees that

$$\Theta_{00}^{(O)}(e^{i\theta}|q) = \Theta_3^{(A)}\left(\frac{\theta}{2}|\sqrt{q}\right). \quad (3.13)$$

Therefore, one can continue the computation and write

$$\widetilde{Z}_{\rm CS} = \frac{(-1)^{N(N-1)/2}N!}{|W|}\left\langle\left(\det_{1\le i,j\le N}((-1)^{j-1}q^{-(j-1)/2}\sqrt{[q]_{j-1}}p_{j-1}(z_i))\right)^2\right\rangle_w, \quad (3.14)$$

which then connects with the usual expression of the partition function in terms of the orthogonal polynomials for the measure on the real line.

Incidentally, both Stieltjes-Wigert and Rogers-Szegö can be interpreted as the ground-state wave function of a q-deformed harmonic oscillator.[33] This is an appealing property as it has been recently shown that the Stieltjes-Wigert polynomial describes B-brane amplitudes on the conifold.[29]

IV. QUANTUM DIMENSIONS AS AVERAGES OF SCHUR POLYNOMIALS IN THE STIELTJES-WIGERT ENSEMBLE

In this section we prove a formula for the averages of Schur polynomials that appears in Refs. 2 and 36, without relying on the equivalence with Chern-Simons theory.

Recall that Schur polynomials $\mathfrak{s}_\lambda$ (Ref. 35) constitute a basis of symmetric functions in a given set of variables $x=(x_i)$ and are indexed by Young's diagrams λ. If the variables x are seen as eigenvalues of some matrix $M \in sl_n$ then $\mathfrak{s}_\lambda(M)\equiv{\rm Tr}_\lambda(M)$ is the trace of M in the representation associated with λ. The Schur polynomials may also be more directly defined in terms of the skew-symmetric polynomials $\mathfrak{a}_\mu=\det(x_i^{\mu_j+n-j})$ as

$$\mathfrak{s}_\lambda(x) \equiv \frac{\mathfrak{a}_{\lambda+\delta}(x)}{\mathfrak{a}_\delta(x)}, \quad (4.1)$$

where $\mathfrak{a}_\delta(x)$ is the Vandermonde in the variables x. The result we want to show is the following:

$$\langle\mathfrak{s}_\lambda(M)\rangle_w = q^{-n|\lambda|-(1/2)C_\lambda^{U(n)}}\mathcal{D}_\lambda, \quad (4.2)$$

with

$$C_\lambda^{U(n)} = (n+1)|\lambda| + \sum_i(\lambda_i^2 - 2i\lambda_i), \quad (4.3)$$

the Casimir of the representation labeled by Young's diagram λ and $|\lambda|$ its total number of boxes. Note that this quantity can be rewritten using Eq. (B6) as

$$C_\lambda^{U(n)} = n|\lambda| + 2(n(\lambda') - n(\lambda)), \tag{4.4}$$

where λ' denotes the conjugate partition. The quantum dimension is defined by the q-hook formula (see, for instance, Sec. 4.4 in Ref. 37 for a clear discussion of the definition and properties of quantum dimensions)

$$\mathcal{D}_\lambda \equiv \prod_{x\in\lambda} \frac{\lfloor n + c(x)\rfloor}{\lfloor h(x)\rfloor}, \tag{4.5}$$

where for each box $x=(i,j)$ of the diagram $h(x) \equiv \lambda_i + \lambda_j' - i - j + 1$ is the hook-length and $c(x) \equiv j - i$ the content of x.

Case of 1-column diagrams. The quantum dimension of the jth fundamental representation of sl_n, which is associated (we will freely switch notations between Young's diagrams and partitions in the following) with the partition (1^j), or a one-column Young tableau of length j, is

$$\mathcal{D}_{(1^j)} = \dim_q \Lambda_{(j)} = \left\lfloor \begin{matrix} n \\ j \end{matrix} \right\rfloor_q. \tag{4.6}$$

Moreover, the monic Stieltjes-Wigert polynomials can be written as (the measure being here normalized such that $\langle 1 \rangle_w = 1$)

$$\pi_n(x) = \sum_{j=0}^{n} (-1)^{n-j} q^{(j-n)(j+n+1/2)} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j = \langle \det(x - M)\rangle_w. \tag{4.7}$$

Besides, the following formula holds for the characteristic polynomial:

$$\det(x - M) = \sum_{j=0}^{n} (-1)^{n-j} \mathfrak{s}_{(1^{n-j})}(M) x^j, \tag{4.8}$$

with $\mathfrak{s}_\lambda(M)$ the Schur polynomial associated with the partition λ. Therefore,

$$\sum_{j=0}^{n} (-1)^{n-j} \langle \mathfrak{s}_{(1^{n-j})}(M)\rangle_w x^j = \sum_{j=0}^{n} (-1)^{n-j} q^{(j-n)(j+n+1/2)} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j, \tag{4.9}$$

from which we extract

$$\langle \mathfrak{s}_{(1^j)}(M)\rangle_w = q^{-j(2n-j+1/2)} \begin{bmatrix} n \\ j \end{bmatrix}_q = q^{-j/2(3n-j+1)} \left\lfloor \begin{matrix} n \\ j \end{matrix} \right\rfloor_q. \tag{4.10}$$

Using Eq. (4.3), one sees that Eq. (4.10) is indeed consistent with Eq. (4.2).

General case. To study the case of general Young's diagram we note that as a generalization of Eq. (4.8), higher powers of the characteristic polynomial are generating functions for diagrams with a higher number of columns. Relying on a formula first computed in Ref. 38 we then relate the average of Schur polynomials to some determinant of Stieltjes-Wigert polynomials. From Ref. 35 (Sec. I.4, example 5, p. 67) we see (taking a slightly more convenient notation)

$$\prod_{i=1}^{k} \prod_{j=1}^{n} (x_i + y_j) = \sum_{\lambda;\lambda_1 \leq k} \mathfrak{s}_\lambda(y) \mathfrak{s}_{\tilde{\lambda}'}(x), \tag{4.11}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is Young's diagram with at most k columns as imposed by the condition $\lambda_1 \leq k$. The associate diagram $\tilde{\lambda}'$ is defined as $(n-\lambda_n, \ldots, n-\lambda_1)$.

Therefore, if one considers $-y_j$ to be the eigenvalues of M, one immediately gets

$$\prod_{i=1}^{k} \det(x_i - M) = \sum_{\lambda;\lambda_1 \leq k} (-1)^{|\lambda|} \mathfrak{s}_\lambda(M) \mathfrak{s}_{\tilde{\lambda}}(x). \tag{4.12}$$

By a standard result on characteristic polynomials,[38] we have

$$\left\langle \prod_{i=1}^{k} \det(x_i - M) \right\rangle_w = \frac{1}{\mathfrak{a}_\delta(x)} \begin{vmatrix} \pi_n(x_1) & \dots & \pi_{n+k-1}(x_1) \\ \vdots & \vdots & \vdots \\ \pi_n(x_k) & \dots & \pi_{n+k-1}(x_k) \end{vmatrix}, \tag{4.13}$$

with $a_\delta(x)$ the Vandermonde determinant of the x variables.

We now turn to the rhs, which we call Δ for convenience. From the explicit expression for the Stieltjes-Wigert polynomials one obtains

$$\begin{aligned} \mathfrak{a}_\delta(x)\Delta &= \sum_{i_1,\dots,i_k} \sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma)(-1)^{i_1+\cdots+i_k} \prod_{j=1}^{k} q^{-i_j(2n+2\sigma(j)-2-i_j+1/2)} \prod_{j=1}^{k} \begin{bmatrix} n+\sigma(j)-1 \\ i_j \end{bmatrix} x_j^{n+\sigma(j)-1-i_j} \\ &= \sum_{i_1,\dots,i_k} (-1)^{i_1+\cdots+i_k+k(k-1)/2} \prod_{j=1}^{k} q^{-i_j(2n-i_j+1/2)-(j-1)(2n+j-1/2)} \\ &\quad \times \left(\sum_{\sigma \in \mathfrak{S}_k} \epsilon(\sigma) \prod_{j=1}^{k} \begin{bmatrix} n+\sigma(j)-1 \\ i_j+\sigma(j)-1 \end{bmatrix} \right) \prod_{j=1}^{k} x_j^{n-i_j} \\ &= \sum_{i_1,\dots,i_k} (-1)^{i_1+\cdots+i_k+k(k-1)/2} \prod_{j=1}^{k} q^{-i_j(2n-i_j+1/2)-(j-1)(2n+j-1/2)} \det_{1\leq a,b\leq k} \left(\begin{bmatrix} n+b-1 \\ i_a+b-1 \end{bmatrix} \right) \prod_{j=1}^{k} x_j^{n-i_j}, \end{aligned}$$

in the second line we have relabeled $i_j \to i_j + \sigma(j) - 1$. Now we study the determinant in the previous expression and show that if $i_1 > \cdots > i_n$,

$$\det_{1\leq a,b\leq k} \left(\begin{bmatrix} n+b-1 \\ i_a+b-1 \end{bmatrix} \right) = A_q(\lambda) \begin{bmatrix} n \\ \lambda \end{bmatrix}, \tag{4.14}$$

with the constant $A_q(\lambda) = q^{n(\lambda')}$, as we show in Appendix B. Here λ' is the partition conjugate to λ and equal to $(i_1, i_2+1, \dots, i_k+k-1)$. $\left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right]$ is a notation generalizing the q-binomial coefficients $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]$, which is defined by the q-hook formula (we warn the reader that for convenience we adopt a slightly different notation for $\left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right]$ compared to Ref. 35 in the sense that its value for the partition (1^r) is the usual Gaussian polynomial $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$, whereas in Ref. 35, $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ (r) \end{smallmatrix}\right]$)

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} \equiv \prod_{x \in \lambda} \frac{1-q^{n+c(x)}}{1-q^{h(x)}}. \tag{4.15}$$

This is nothing but the analog of quantum dimension Eq. (4.5) where instead of using the $[\cdot]$ version of the q-integers one rather uses $\lfloor\cdot\rfloor$ (recall that $[n] = q^{(n-1)/2} \lfloor n \rfloor$). Identifying with the lhs of Eq. (4.12) we obtain

$$\langle \mathfrak{s}_\lambda(M) \rangle_w = q^{\sum_j -i_j(2n-i_j+1/2)-(j-1)(2n+j-1/2)} q^{n(\lambda')} \begin{bmatrix} n \\ \lambda \end{bmatrix}. \tag{4.16}$$

The last step we need to perform now is to convert $\left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right]$ in terms of $\mathcal{D}_\lambda$ and check that the prefactor is given by Eq. (4.3). To this end we first note that due to Eqs. (B8) and (B7) we have

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = q^{1/2(n-1)|\lambda| - n(\lambda)} \mathcal{D}_\lambda \tag{4.17}$$

and

$$\sum_j -i_j\left(2n-i_j+\frac{1}{2}\right)-(j-1)\left(2n+j-\frac{1}{2}\right)=-\left(2n-\frac{3}{2}\right)|\lambda|+\sum_j \lambda_j'^2-2j\lambda_j'. \quad (4.18)$$

To rewrite things in terms of the partition itself, rather than its transposed, we use the relationship

$$\sum_i \lambda_i^2-2i\lambda_i=2(n(\lambda')-n(\lambda))-|\lambda|. \quad (4.19)$$

Collecting all the prefactors we can eventually write our final result

$$\langle \mathfrak{s}_\lambda\rangle = q^{-1/2((3n+1)|\lambda|+\Sigma_i\lambda_i^2-2i\lambda_i)}\mathcal{D}_\lambda, \quad (4.20)$$

which coincides with Eq. (4.2).

V. CONCLUSIONS AND OUTLOOK

We have constructed the biorthogonal Stieltjes-Wigert polynomials, necessary for computing expressions such as Eq. (2.1), which appear in Chern-Simons theory. The polynomials are not discussed in the mathematics literature, so a great deal of effort has been devoted to the explicit description of their fundamental properties.

The construction of the biorthogonal Stieltjes-Wigert polynomials may very well be a necessary technical step for the computation of knot invariants in exact fashion employing orthogonal polynomials. Note that, so far, the topological invariants computed with orthogonal polynomials are only Chern-Simons partition functions [more precisely, only the case of S^3 with gauge group $U(N)$].[8] Indeed, according to Mariño,[39] the results in Ref. 4 can be extended in order to obtain random matrix descriptions of other Chern-Simons observables. The case of torus knots, for example, amounts to

$$W_R^{(P,Q)}=\frac{\mathrm{e}^{-g_s/2(PQ(\Lambda^2-\rho^2)+(P/Q+Q/P)\rho^2)}}{|PQ|^{N/2}N!}\int\prod_{i=1}^N\frac{du_i}{2\pi}\mathrm{e}^{-\Sigma_i u_i^2/2g_s}\prod_{i<j}\left(2\sinh\frac{u_i-u_j}{2P}\right)\left(2\sinh\frac{u_i-u_j}{2Q}\right)S_\lambda(\mathrm{e}^{u_i}), \quad (5.1)$$

where $S_\lambda(x_i)$ are Schur polynomials associated to the partition λ [representations of $U(N)$ are labeled by partitions λ].

That is to say, torus knots correspond to Eq. (2.1) studied here, but with an insertion of a Schur polynomial. However, an obstacle could be the lack of a computational device for random matrixlike quantities with such a term. Note that the case of an ordinary Gaussian Hermitian matrix model with a Schur polynomial insertion was solved in Ref. 40 by purely combinatorial methods, with no use of Hermite polynomials at all. Nevertheless, as we have seen in the last section, a computation of the Stieltjes-Wigert ensemble with the Schur polynomial can be carried out with a mixture of combinatorial and orthogonal polynomial techniques. Therefore, it turns out that we have studied in detail the two cases comprised in Eq. (5.1): the biorthogonal case without the Schur insertion and the average of the Schur polynomial in the orthogonal ($P=Q=1$) ensemble.

We have also studied other aspects of the (ordinary) Stieltjes-Wigert polynomials that are of direct relevance for the corresponding matrix model. In particular, we have discussed the close ties with Rogers-Szegö polynomials, which are defined on the unit circle (both sets of polynomials being an equivalent solution of a q-deformed harmonic oscillator problem). This relationship allows us to clearly establish the relationship with the unitary matrix model discussed in Ref. 15. Fundamental properties of the Stieltjes-Wigert polynomials such as their asymptotic behavior and the above mentioned q-deformed harmonic oscillator property may be of interest in connection with the recently established role of the polynomials in the study of topological strings.[29] We hope to address some of these issues in future work.

ACKNOWLEDGMENTS

The authors are grateful to Edouard Brézin, Bernard Julia, Vladimir Kazakov, and Marcos Mariño for valuable comments and correspondence.

APPENDIX A: NORMALIZATIONS

To follow standard conventions it is convenient to have the orthogonal polynomials either monic or normalized to unity. Hence, we rewrite the previous ones a little bit (it will also make the link with the usual Stieltjes-Wigert for $k=1$ more transparent).

1. Notations: $k=1$

From Szegö[27] we have for the Stieltjes-Wigert polynomials

$$p_n(x) = \frac{(-1)^n q^{n/2+1/4}}{\sqrt{[q]_n}} \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_q q^{\nu^2}(-\sqrt{q}x)^{\nu}, \tag{A1}$$

with $\left[\begin{smallmatrix} n \\ \nu \end{smallmatrix}\right]_q$ the q-binomial coefficient

$$\begin{bmatrix} n \\ \nu \end{bmatrix}_q \equiv \frac{[q]_n}{[q]_\nu [q]_{n-\nu}}.$$

These polynomials are orthonormal for the scalar product $\langle , \rangle_w$ induced by

$$w(x) = \frac{\kappa}{\sqrt{\pi}} e^{-\kappa^2 \log^2 x}, \tag{A2}$$

with $q=e^{-1/2\kappa^2}$ as usual. Note that one has

$$\langle 1,1 \rangle_w = 1/\sqrt{q}. \tag{A3}$$

The S_n polynomials in Eq. (2.8) to which Askey refers[25] as the SW polynomials are written in a slightly different form. They satisfy

$$\langle S_n, S_m \rangle = \frac{q^{-n}}{[q]_n} \delta_{n,m}, \tag{A4}$$

with $\langle , \rangle$ the scalar product associated with the measure $A dx/[-x]_\infty[-q/x]_\infty$ with A a normalization constant such that $\langle 1,1 \rangle = 1$.

Then, the polynomials defined by

$$\tilde{S}_n(x) \equiv (-1)^n \sqrt{[q]_n} q^{n/2} S_n(x) \tag{A5}$$

are orthonormal for $\langle , \rangle$. One then sees that

$$p_n(x) = q^{1/4} \tilde{S}_n(\sqrt{q}x). \tag{A6}$$

Since Al-Salam and Verma[26] have the same notations as Askey for the q-Laguerre polynomials, we will make the same rewriting when using the biorthogonal Stieltjes-Wigert polynomials in the context of Chern-Simons theory computations.

2. *k* arbitrary

Normalizing and changing variables as in the previous section we define new polynomials

$$R_n(x,k|q) \equiv \frac{(-1)^n q^{1/4}}{\sqrt{k_n}} Y_n(\sqrt{q}x,k|q) = r_{n,k}x^n + \ldots \tag{A7}$$

and

$$T_n(x,k|q) \equiv \frac{(-1)^n q^{1/4}}{\sqrt{k_n}} Z_n(\sqrt{q}x,k|q) = t_{n,k}x^{nk} + \ \ldots . \tag{A8}$$

Then one has

$$r_{n,k} = \frac{q^{(n+1/2)^2}}{\sqrt{[q]_n}} q^{(k-1)n^2/2} \tag{A9}$$

and

$$t_{n,k} = \frac{\sqrt{[q]_n}}{(q^k;q^k)_n} q^{(nk+1/2)^2 - 1/2[n^2k(k-1)]}, \tag{A10}$$

which reduce to $q^{(n+1/2)^2}/\sqrt{[q]_n}$ when $k=1$ as expected. Then one has

$$\langle R_n(x,k|q) T_n(x,k|q)\rangle_w = \delta_{m,n}. \tag{A11}$$

Following Eqs. (2.48) and (2.53) the recurrence relations for these orthonormal polynomials read

$$xR_n(x,k|q) = q^{-nk-1/2}(R_n(q^{-1}x,k|q) + q^{-k/2}\sqrt{1-q^{n+1}}R_{n+1}(q^{-1}x,k|q)) \tag{A12}$$

$$x^k T_n(x,k|q) = q^{-nk+[k(k-2)/2]}\left(T_n(q^{-k}x,k|q) + q^{-k/2}(1-q^{k(n+1)}) \sqrt{\frac{k_{n+1}}{k_n}} T_{n+1}(q^{-k}x,k|q)\right). \tag{A13}$$

APPENDIX B: PROOF OF EQUATION (4.14)

Equation (4.14) is not obvious at first sight because when $q=1$, Giambelli's formula (see, for instance, Ref. 41, Eq. (16.114) for a nice presentation) would lead us to write

$$\det_{1\le a,b\le k}\left(\binom{n}{i_a+b-1} \right) = \dim \lambda . \tag{B1}$$

However, classically one also has

$$\det_{1\le a,b\le k}\left(\binom{n}{i_a+b-1} \right) = \det_{1\le a,b\le k}\left(\binom{n+b-1}{i_a+b-1} \right), \tag{B2}$$

which can be seen to hold by using Pascal's identity. Nevertheless, Pascal's identity in the quantum case is slightly more complicated and reads

$$\begin{bmatrix} n+1 \\ j+1 \end{bmatrix} = \begin{bmatrix} n \\ j+1 \end{bmatrix} + q^{n-j}\begin{bmatrix} n \\ j \end{bmatrix}, \tag{B3}$$

and we thus see that in the quantum case the same kind of simplification cannot be shown to hold as simply as in the classical setting.

To prove Eq. (4.14) nevertheless, first recall that in the space of symmetric polynomials the change of basis between the elementary symmetric polynomials (elementary symmetric polynomials are special cases of Schur polynomials of 1-column diagrams, or $e_r = \mathfrak{s}_{\Lambda_r}$ in our notations) e_r and the Schur polynomials $\mathfrak{s}_\lambda$ is given by [Ref. 35, Eq. (3.5)]

$$\mathfrak{s}_\lambda = \det(e_{\lambda_i' - i + j}), \tag{B4}$$

where $\lambda = (\lambda_1 \geqslant \cdots \geqslant l_n)$ is a partition and λ' its conjugate partition. This is actually nothing else than Giambelli's identity, written for the symmetric polynomials and not just for the dimensions of the associated representations. To compute from this, note that if one considers $x = (1, q, \ldots, q^{n-1})$, then one has (Ref. 35, Sec. I.3, example 1, p. 44)

$$\mathfrak{s}_\lambda(x) = q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix}, \tag{B5}$$

where $n(\lambda)$ is defined as

$$n(\lambda) \equiv \sum_{i \geqslant 1} (i-1)\lambda_i = \sum_{j \geqslant 1} \binom{\lambda'}{2}, \tag{B6}$$

and satisfies the following useful formulas:

$$\sum_{x \in \lambda} c(x) = n(\lambda') - n(\lambda) \tag{B7}$$

for the content (Ref. 35, Sec. 1, Example 3, p. 11), and another one for the hook lengths (Ref. 35, Sec. 1, Example 2, p. 11),

$$\sum_{x \in \lambda} h(x) = n(\lambda) + n(\lambda') + |\lambda|. \tag{B8}$$

Let us come back to our computation and particularize Eq. (B4) to $x = (1, q, \ldots, q^{n-1})$ that gives

$$q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix} = \det\left(q^{(\lambda_i' - i + j)(\lambda_i' - i + j - 1)/2} \begin{bmatrix} n \\ \lambda_i' - i + j \end{bmatrix} \right), \tag{B9}$$

or, introducing $i_a + a - 1 = \lambda_a'$,

$$q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix} = \det\left(q^{(i_a + b - 1)(i_a + b - 2)/2} \begin{bmatrix} n \\ i_a + b - 1 \end{bmatrix} \right). \tag{B10}$$

This is still not quite what we want. To proceed further note that according to Eq. (B3) one has

$$q^{j(j+1)/2} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} = q^{j(j+1)/2} \begin{bmatrix} n \\ j+1 \end{bmatrix} + q^n \left(q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix} \right). \tag{B11}$$

Therefore, by multiple linear combinations of columns one can write

$$\det\left(q^{(i_a + b - 1)(i_a + b - 2)/2} \begin{bmatrix} n \\ i_a + b - 1 \end{bmatrix} \right) = \det\left(q^{(i_a + b - 1)(i_a + b - 2)/2} \begin{bmatrix} n + b - 1 \\ i_a + b - 1 \end{bmatrix} \right). \tag{B12}$$

Now, for convenience extract a factor $q^{i_a(i_a - 1)/2}$ in each line to get

$$\det\left(q^{(i_a+b-1)(i_a+b-2)/2}\begin{bmatrix} n+b-1 \\ i_a+b-1 \end{bmatrix}\right) = q^{\Sigma_a(i_a(i_a-1)/2)}\det\left(q^{(b-1)(2i_a+b-2)/2}\begin{bmatrix} n+b-1 \\ i_a+b-1 \end{bmatrix}\right). \tag{B13}$$

To proceed further note the following property of the q-binomial coefficients:

$$q^{i_a+b}\begin{bmatrix} n+b \\ i_a+b \end{bmatrix} = \begin{bmatrix} n+b \\ i_a+b \end{bmatrix} + (q^{n+b}-1)\begin{bmatrix} n+b-1 \\ i_a+b-1 \end{bmatrix}. \tag{B14}$$

Therefore, once we write

$$q^{(i_a+b)(i_a+b-1)/2}\begin{bmatrix} n+b \\ i_a+b \end{bmatrix} = q^{-b(b+1)/2}q^{b(i_a+b)}\begin{bmatrix} n+b \\ i_a+b \end{bmatrix}, \tag{B15}$$

it is easy to see that

$$\det\left(q^{(b-1)(2i_a+b-2)/2}\begin{bmatrix} n+b-1 \\ i_a+b-1 \end{bmatrix}\right) = q^{-\Sigma_j j(j-1)/2}\det\left(\begin{bmatrix} n+b-1 \\ i_a+b-1 \end{bmatrix}\right). \tag{B16}$$

Collecting everything we thus have

$$\det\left(\begin{bmatrix} n+b-1 \\ i_a+b-1 \end{bmatrix}\right) = q^{\Sigma_j[j(j-1)/2]-\Sigma_j[i_j(i_j-1)/2]+n(\lambda)}\begin{bmatrix} n \\ \lambda \end{bmatrix}.$$

From which we finally obtain

$$A_q(\lambda) = q^{n(\lambda')}. \tag{B17}$$

APPENDIX C: NOTATION FOR CHERN-SIMONS QUANTITIES

We give here some information about the Chern-Simons quantities that appear in the text, mainly in Eq. (1.3). For more information, see Ref. 4 and references therein. To understand the origin of other quantities in Eq. (1.3), one has to take into account the constructions of Seifert homology spheres from surgery. Seifert homology spheres can be constructed by performing surgery on a link $\mathfrak{L}$ in $\mathbf{S}^3$ with $n+1$ components, consisting on n parallel and unlinked unknots together with a single unknot whose linking number with each of the other n unknots is one. The surgery data are p_j/q_j for the unlinked unknots, $j=1,\ldots,n$, and 0 on the final component. p_j is coprime to q_j for all $j=1,\ldots,n$, and the p_j's are pairwise coprime. After doing surgery, one obtains the Seifert space $M=X(p_1/q_1,\ldots,p_n/q_n)$. This is rational homology sphere whose first homology group $H_1(M,\mathbf{Z})$ has order $|H|$, where

$$H = P\sum_{j=1}^{n}\frac{q_j}{p_j} \quad \text{and} \quad P = \prod_{j=1}^{n} p_j. \tag{C1}$$

Another topological invariant that will enter the computation is the signature of $\mathfrak{L}$, which turns out to be

$$\sigma(\mathcal{L}) = \sum_{i=1}^{n}\operatorname{sgn}\left(\frac{q_i}{p_i}\right) - \operatorname{sgn}\left(\frac{H}{P}\right). \tag{C2}$$

For $n=1,2$, Seifert homology spheres reduce to lens spaces, and one has that $L(p,q)=X(q/p)$. For $n=3$, we obtain the Brieskorn homology spheres $\Sigma(p_1,p_2,p_3)$ (in this case the manifold is independent of q_1,q_2,q_3). In particular, $\Sigma(2,3,5)$ is the Poincaré homology sphere. Finally, the Seifert manifold $X[2/-1,m/(m+1)/2,(t-m)/1]$, with m odd, can be obtained by integer surgery on a

$(2,m)$ torus knot with framing t. Note that in Eq. (1.3) the weight and root lattices of G are denoted by Λ_{w} and Λ_{r}, respectively.

Finally, there is a phase factor in Eq. (1.3) that comes from the framing correction, which guarantees that the resulting invariant is in the canonical framing for the three-manifold M. Its explicit expression is

$$\phi = 3\ \mathrm{sgn}\left(\frac{H}{P}\right) + \sum_{i=1}^{n} 12 s(q_i,p_i) - \frac{q_i}{p_i}, \tag{C3}$$

where $\sigma(\mathcal{L})$ is again the signature of the linking matrix of $\mathcal{L}$ and $s(p,q)$ is the Dedekind sum,

$$s(p,q) = \frac{1}{4q}\sum_{n=1}^{q-1} \cot\left(\frac{\pi n}{q}\right)\cot\left(\frac{\pi np}{q}\right). \tag{C4}$$

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