1. Anomalies of zeta regularized determinants come and go

It is well known that the study of zeta functions is central for the issue of giving a sense to the definition of determinant of a pseudodifferential operator (ΨDO). This definition goes back to Ray and Singer: for an operator A with spectrum $\lambda_i, i \in I$ (here I needs not be discrete, it can be a 'multiindex' made up of parts of different nature), formally

$$\det A = \prod_{i \in I} \lambda_i = \exp\left(\sum_{i \in I} \log \lambda_i\right). \tag{1.1}$$

But from the definition of the zeta function

$$\zeta_A(s) = \sum_{i \in I} \lambda_i^{-s}, \tag{1.2}$$

it turns out that

$$\zeta_A'(0) = -\sum_{i \in I} \log \lambda_i. \tag{1.3}$$

It is most natural then to define (as Ray and Singer did) the determinant of A by means of the zeta function as

$$\det\nolimits_\zeta A \equiv \exp\left[-\zeta_A{}'(0)\right], \tag{1.4}$$

Note that this is a *definition*, since the above manipulations are formal as long as the convergence properties of the expressions at hand are not fully specified, in accordance with the theorem at the beginning of this section. This is taken care of by the analytical continuation provided in the definition of the zeta function of A.

Now, it would seem clear that, if we have a product of two commuting operators,

$$\begin{aligned} \det\nolimits_\zeta(AB) &= \exp\left[\sum_{i \in I} \log(\lambda_i \mu_i)\right] = \exp\left[\sum_{i \in I} (\log \lambda_i + \log \mu_i)\right] \\ &= \exp\left[\sum_{i \in I} \log \lambda_i + \sum_{i \in I} \log \mu_i\right] = \exp\left[\sum_{i \in I} \log \lambda_i\right] \exp\left[\sum_{i \in I} \log \mu_i\right] \\ &= \det\nolimits_\zeta A \, \det\nolimits_\zeta B. \end{aligned} \tag{1.5}$$

But this is *not* true. Below we provide some specific examples.

Very much related with this is the fact that the zeta function trace

$$\mathrm{tr}_\zeta A = \sum_{i \in I} \lambda_i = \zeta_A(s = -1) \tag{1.6}$$

fails to satisfy the additive property: in general

$$\mathrm{tr}_\zeta (A + B) \neq \mathrm{tr}_\zeta A + \mathrm{tr}_\zeta B, \tag{1.7}$$

for, again, this is a regularized trace (involves analytical continuation) which is used with non trace-class operators.

As an example, consider the following commuting linear operators in an infinite-dimensional space, given in diagonal form by:

$$A_1 = \text{diag }(1, 2, 3, 4, \ldots), \qquad A_2 = \text{diag }(1, 1, 1, 1, \ldots), \tag{1.8}$$

and their sum

$$A_1 + A_2 = \text{diag }(2, 3, 4, 5, \ldots). \tag{1.9}$$

The corresponding ζ-traces are easily obtained:

$$\begin{aligned} \text{tr}_\zeta A_1 = \zeta_R(-1) = -\frac{1}{12}, \qquad \text{tr}_\zeta A_2 = \zeta_R(0) = -\frac{1}{2}, \\ \text{tr}_\zeta (A_1 + A_2) = \zeta_R(-1) - 1 = -\frac{13}{12}, \end{aligned} \tag{1.10}$$

ζ_R being the Riemann zeta function. The last trace has been calculated according to the rules of infinite series summation. We observe that

$$\text{tr}_\zeta (A_1 + A_2) - \text{tr}_\zeta A_1 - \text{tr}_\zeta A_2 = -\frac{1}{2} \neq 0. \tag{1.11}$$

Unlike for ordinary, finite dimensional determinants, for which we have the property: $\det(AB) = \det(A)\det(B)$, for zeta determinants one rather has to consider, in general, an additional piece. It is usually written as

$$a(A, B) = \ln \frac{\det(AB)}{\det(A)\det(B)} \tag{1.12}$$

or

$$a\,(A, B) = \zeta_A'\,(0) + \zeta_B'\,(0) - \zeta_{AB}'\,(0)\,. \tag{1.13}$$

Thus the anomaly $a(A, B)$ will vanish if the derivatives at $s = 0$ of the respective zeta function satisfy the additive property. There is an explicit expression, due to Wodzicki, for $a\,(A, B)$, that simplifies enormously the calculation of the multiplicative anomaly in many cases.

There are many examples of simple cases with and without multiplicative anomaly. We give now a condition that guarantees its absence. Consider the two following zeta functions:

$$\zeta_A\,(s) = \sum_i \lambda_i^{-s}, \tag{1.14}$$

$$\zeta_B\,(s) = \sum_i (c\lambda_i^\alpha)^{-s} = c^{-s}\zeta_A\,(\alpha s)\,, \text{ with } c, \alpha \in \mathbf{R}. \tag{1.15}$$

The zeta function associated with the product of the eigenvalues is

$$\zeta_{AB}\,(s) = \sum_i \left(c\lambda_i^{\alpha+1}\right)^{-s} = c^{-s}\zeta_A\,((\alpha+1)\,s)\,, \tag{1.16}$$

and thus

$$\zeta_{AB}(s) = c^{-s}\zeta_A\,((\alpha+1)\,s) \neq \zeta_A\,(s) + c^{-s}\zeta_A\,(\alpha s) \tag{1.17}$$

Taking the derivative and performing the substitution $s = 0$, we have that:

$$\zeta_{AB}'\,(0) = -\ln c\zeta_A(0) + (\alpha+1)\zeta_A'(0) = \zeta_A'\,(0) + \zeta_B'\,(0)\,. \tag{1.18}$$

Therefore, in spite of the fact that the two zeta functions are different, their respective derivatives at zero are equal. This is enough to guarantee the absence of the multiplicative anomaly, namely $a\,(A,B)=0$. This is quite a general situation, since we have not fixed the λ_i at all. We have only played with the relative difference between the spectra.

On the other hand, a rather different thing is to consider two spectra which are related by an *additive* constant:

$$\mu_i = \lambda_i + c. \tag{1.19}$$

For simplicity, let us restrict our analysis to the specific example

$$\lambda_n = n, \qquad \mu_n = n+1, \qquad n = 1,2,3,... \tag{1.20}$$

Thus

$$\zeta_A(s) = \zeta_R(s), \qquad \zeta_B(s) = \zeta_R(s) - 1, \tag{1.21}$$

while the zeta function of the product is of Epstein type:

$$\zeta_A(s) = \sum_{n=1}^{\infty} (n^2+n)^{-s} = \sum_{n=1}^{\infty} \frac{\Gamma(n+s)\,2^{-2n}}{n!\,\Gamma(s)}\,\zeta_H(2(n+s),3/2). \tag{1.22}$$

Thus

$$\zeta_A'(0) + \zeta_B'(0) = 2\zeta_R'(0) = -\ln(2\pi), \tag{1.23}$$

while

$$\zeta_{AB}'(0) = \sum_{n=1}^{\infty} \frac{2^{-2n}}{n}\,\zeta_H(2n,3/2), \tag{1.24}$$

which are not equal. Numerically

$$\zeta_{AB}'(0) = 0.4417, \qquad \zeta_A'(0) + \zeta_B'(0) = -1.8379, \tag{1.25}$$

even the signs are different and the anomaly, in such a simple case, is larger in absolute value than the individual results themselves:

$$a = \zeta_A'(0) + \zeta_B'(0) - \zeta_{AB}'(0) = -2.2796. \tag{1.26}$$