

1. THE RANDOM MATRIX MODEL THAT PROVIDES THE CLOSEST CONNECTION WITH GAUGE THEORY IS THE STIELTJES-WIGERT ENSEMBLE.

This is simple to prove. Consider the Hermitian random matrix ensemble:

$$(1.1) \quad P(x_1, \dots, x_N) = C_N \prod_{i < j} |x_i - x_j|^2 \prod_{i=1}^N e^{-V(x_i)},$$

Instead of the very much studied Gaussian $V(x) = x^2$ or polynomial $V(x) = \sum_n a_n x^n$ cases, consider the log-normal case:

$$(1.2) \quad \omega(x) = e^{-V(x)} = \frac{1}{\sqrt{\pi}} k e^{-k^2 \log^2 x},$$

the ensemble is named Stieltjes-Wigert, due to the name of the associated orthogonal polynomials. Actually, this weight and its associated orthogonal polynomials have a distinguished story and many remarkable properties as well.

For the moment, let us just compute the partition function:

$$(1.3) \quad Z = \int \prod_{i=1}^N \frac{1}{\sqrt{\pi}} k e^{-k^2 \log^2 x_i} \prod_{i < j} |x_i - x_j|^2$$

We do this in detail:

For this, we can use the following well-known result for the partition function in terms of the orthogonal polynomials :

$$(1.4) \quad Z = \int \dots \int \prod_{i=1}^N \omega(x_i) dx_i \prod_{i < l} (x_i - x_l)^2 = \frac{N!}{\prod_{i=0}^{N-1} a_i^2} = N! a_0^{-2N} \prod_{i=1}^{N-1} \left(\left(\frac{a_{i-1}}{a_i} \right)^2 \right)^{N-i}.$$

where the coefficient a_i is:

$$(1.5) \quad p_i(x) = a_i x^i + \dots$$

The first step is to compute the Z associated to the Stieltjes-Wigert orthogonal polynomials [?]. The coefficients are given by:

$$(1.6) \quad a_j = q^{(j+1/2)^2} \{(1-q) \dots (1-q^j)\}^{-1/2},$$

then

$$(1.7) \quad \left(\frac{a_{j-1}}{a_j} \right)^2 = q^{-4j} (1-q^j),$$

and $a_0 = q^{1/4}$. Therefore:

$$(1.8) \quad \begin{aligned} Z_{sw} &= N! q^{-N/2} \prod_{j=1}^{N-1} q^{-4j(N-j)} (1-q^j)^{N-j} \\ &= N! q^{-\frac{1}{8}N(2N-1)(2N+1)} \prod_{j=1}^{N-1} (1-q^j)^{N-j}, \end{aligned}$$

Interestingly enough, it is known that the partition function of Chern-Simons theory on S^3 with $U(N)$ is essentially given by:

$$(1.9) \quad Z = \frac{\exp\left(-\frac{g_s}{12}N(N^2-1)\right)}{N!} \int \prod_{i<j} \left(2 \sinh \frac{u_i - u_j}{2}\right)^2 \prod_{i=1}^N e^{-u_i^2/2g_s} \frac{du_i}{2\pi},$$

Happily enough, these two models are simply related by:

$$u_i = \log x_i + Ng_s,$$

Actually a mapping such as $u_i = \log x_i$ also works, due to rather nice properties of the Stieltjes-Wigert ensemble. Thus:

$$(1.10) \quad \int \prod_{i=1}^N \frac{du_i}{2\pi} \exp\left(-\frac{u_i^2}{2g_s}\right) \prod_{i<j} \left(2 \sinh \left(\frac{u_i - u_j}{2}\right)\right)^2 \\ = (2\pi)^{-N} \exp\left(\frac{-N^3 g_s}{2}\right) \int \prod_{i=1}^N dx_i \exp\left(-\frac{\log^2(x_i)}{2g_s}\right) \prod_{i<j} (x_i - x_j)^2.$$

Then:

$$(1.11) \quad \mathcal{Z}(S^3) = \frac{e^{-\frac{1}{12}g_s N(N^2-1)}}{N!} \left(\frac{2\pi}{g_s}\right)^{-N/2} \mathcal{Z}_{sw}$$

Thus the partition function of the Hermitian Stieltjes-Wigert ensemble gives the partition function of a topological quantum field theory in three-dimensions. Namely, Chern-Simons theory on S^3 with gauge group $U(N)$. additionally, it seems that the Stieltjes-Wigert model posses many more remarkable properties in relationship with Chern-Simons theory. To make explicit contact with the typical expressions for the Chern-Simons partition functions we make some transformations on the product term in Z_{sw} :

$$\prod_{j=1}^{N-1} (1 - q^j)^{N-j} = \prod_{j=1}^{N-1} \left(2 \sinh \frac{g_s j}{2}\right)^{N-j} \exp\left(-\frac{g_s j(N-j)}{2}\right) \\ (1.12) \quad = \exp\left(\frac{-1}{12}g_s N(N^2-1)\right) \exp\left(\frac{i\pi}{4}N(N-1)\right) \cdot \prod_{j=1}^{N-1} \left(2 \sin \frac{g_s j}{2i}\right)^{N-j}.$$

The final expression for the partition function is:

$$(1.13) \quad Z = \frac{\exp\left(-\frac{1}{12}g_s N(N^2-1)\right)}{N!} Z_{\sinh} \\ = \left(\frac{g_s}{2\pi}\right)^{N/2} \exp\left(\frac{i\pi}{4}N(N-1)\right) \prod_{j=1}^{N-1} \left(2 \sin \frac{g_s j}{2i}\right)^{N-j},$$

and since $g_s = \frac{2\pi i}{k+N}$, we finally find:

$$(1.14) \quad \mathbf{Z} = e^{\frac{1}{4}i\pi N^2} (k+N)^{-N/2} \prod_{j=1}^{N-1} \left(2 \sin \frac{\pi j}{k+N}\right)^{N-j}.$$