Multiplicative anomaly and zeta factorization

E. Elizalde and M. Tierz
Instituto de Ciencias del Espacio (ICE/CSIC),
Institut d'Estudis Espacials de Catalunya (IEEC/CSIC),
Edifici Nexus, Gran Capità, 2-4, 08034 Barcelona, Spain.

Abstract

Some aspects of the multiplicative anomaly of zeta determinants are investigated. A rather simple approach is adopted and, in particular, the question of zeta function factorization, together with its possible relation with the multiplicative anomaly issue is discussed. We look primordially into the zeta functions instead of the determinants themselves, as was done in previous work. That provides a supplementary view, regarding the appearance of the multiplicative anomaly. Finally, we briefly discuss determinants of zeta functions that are not in the pseudodifferential operator framework.

I. Introduction

A pseudodifferential operator (ΨDO) A of order m on a manifold M_n is defined through its symbol $a(x,\xi)$, which is a function belonging to the space $S^m(\mathbf{R}^n \times \mathbf{R}^n)$ of $\mathbf{C}^\infty$ functions such that for any α, β there exists a constant $C_{\alpha,\beta}$ so that $\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$. The definition of A is given (in the distribution sense) by

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi, \tag{1}$$

where f is a smooth function ($f \in \mathcal{S}$) and $\hat{f}$ its Fourier transform. When $a(x,\xi)$ is a polynomial in ξ one gets a differential operator but, in general, the order m can be even complex. For A a positive-definite elliptic ΨDO of positive order $m \in \mathbf{R}$, acting on the space of smooth sections of an n-dimensional vector bundle E over a closed, n-dimensional manifold M, the zeta function is defined as

$$\zeta_A(s) = \mathrm{tr}\ A^{-s} = \sum_j \lambda_j^{-s}, \qquad \mathrm{Re}\ s > \frac{n}{m} \equiv s_0. \tag{2}$$

Here s_0 is called the abscissa of convergence of $\zeta_A(s)$, which is proven to have a meromorphic continuation to the whole complex plane $\mathbf{C}$ (regular at s_0), provided that A admits a spectral cut: $L_\theta = \{\lambda \in \mathbf{C}; \mathrm{Arg}\,\lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\mathrm{Spec}\, A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition).

The Wodzicki (or noncommutative) residue[1] is the only extension of the Dixmier trace to ΨDOs which are not in $\mathcal{L}^{(1,\infty)}$. Even more, it is the only trace one can define in the algebra of ΨDOs up to a multiplicative constant, and is given by the integral

$$\mathrm{res}\ A = \int_{S^*M} \mathrm{tr}\ a_n(x,\xi)\, d\xi, \tag{3}$$

with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral). If $\dim M = n = -\ \mathrm{ord}\ A$ (M compact Riemann, A elliptic, $n \in \mathbf{N}$) it coincides with the Dixmier trace, and one has[1]

$$\mathrm{Res}_{s=1} \zeta_A(s) = \frac{1}{n}\, \mathrm{res}\ A^{-1}. \tag{4}$$

However, the Wodzicki residue continues to make sense for ΨDOs of arbitrary order and, even if the symbols $a_j(x,\xi)$, $j < m$, are not invariant under coordinate choice, the integral in (3) is, and defines a trace. In particular, the residua of the poles of the extended definition of zeta function to operators of complex order are also given by the noncommutative residue.

It is well known that the study of zeta functions is central —at least at a basic level, the one needed in fact in usual applications to physics[2,3]— for the issue of giving a sense to the definition of determinant of a ΨDO (see[4] for the actual state of the art of this concept). This definition goes back to Ray and Singer[5]: for an operator A with spectrum $\lambda_i, i \in I$ (here I needs not be discrete, it can be a 'multiindex' made up of parts of different nature), formally

$$\det A = \prod_{i\in I} \lambda_i = \exp\left(\sum_{i\in I} \log \lambda_i\right) . \tag{5}$$

But from the definition of the zeta function

$$\zeta_A(s) = \sum_{i\in I} \lambda_i^{-s}, \tag{6}$$

it turns out that

$$\zeta_A'(0) = -\sum_{i\in I} \log \lambda_i. \tag{7}$$

It is most natural then to define (as Ray and Singer did) the determinant of A by means of the zeta function as[5]

$$\det_\zeta A \equiv \exp\left[-\zeta_A{}'(0)\right], \tag{8}$$

Note that this is a *definition*, since the above manipulations are formal as long as the convergence properties of the expressions at hand are not fully specified, in accordance with the theorem at the beginning of this section. This is taken care of by the analytical continuation provided in the definition of the zeta function of A.

The definition of the determinant $\det_\zeta A$ only depends on the homotopy class of the spectral cut for A (see above). And one has the following (very useful) asymptotic expansion for the heat kernel:

$$\mathrm{tr}_{t\downarrow 0}\ e^{-tA} = \sum_{\lambda\in \mathrm{Spec}\, A}{}' e^{-t\lambda} \sim \alpha_n(A) + \sum_{n\neq j\geq 0} \alpha_j(A) t^{-s_j} + \sum_{k\geq 1} \beta_k(A) t^k \ln t, \tag{9}$$

where

$$\alpha_n(A) = \zeta_A(0), \qquad \alpha_j(A) = \Gamma(s_j)\mathrm{Res}_{s=s_j}\,\zeta_A(s),\ s_j \notin -\mathbf{N}, \tag{10}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!}\left[PP\zeta_A(-k) + \psi(k+1)\mathrm{Res}_{s=-k}\,\zeta_A(s)\right],\ s_j = -k,\ k \in \mathbf{N},$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!}\mathrm{Res}_{s=-k}\,\zeta_A(s), \quad k \in \mathbf{N}\backslash\{0\}.$$

This paper is organized as follows: in the next Section we give a short but rather self-contained introduction to the appearance of the multiplicative anomaly[6–8] of zeta determinants. In Sec. III, we point out certain particularities of this anomaly by presenting two very different cases: On the one hand, a new and rather general condition that guarantees the absence of anomaly and, in sharp contrast, a quite particular and very simple case where the anomaly is already non-zero. Then, in Sec. IV, extending and complementing previous work on this subject,[7,8] we pay attention to the product of zeta functions rather than its associated determinants. This lead to the consideration of $\det(B \otimes C)$ instead of $\det(BC)$, with B and C two arbitrary operators. Thus, from that point of view, we are able to obtain rather simple new expressions for determinants, mainly thanks to the strong property of factorization of the zeta function. In the last Section we present, in a somewhat more qualitative way, the relationship between all the previous concepts, multiplicative anomaly and zeta factorization, with the appearance of complex poles in the zeta function and other zeta functions that does not belong to the pseudodifferential operator framework. In the Appendix, due to the implementation in a regularization context, we investigate further the topic of zeta function factorization, presenting results, mainly from Number Theory, with two opposite points of view: the construction of Dirichlet L functions from multiplication of simple zeta functions on one hand, and the decomposition of a zeta function in terms of simpler factors on the other hand. This leads to some physical interpretation for the associated heat kernel that we briefly discuss.

II. Appearance of the multiplicative anomaly

Now, it may seem clear that, if we have a product of two commuting operators,

$$\begin{aligned}
\det_\zeta(AB) &= \exp\left[\sum_{i\in I}\log(\lambda_i\mu_i)\right] = \exp\left[\sum_{i\in I}(\log\lambda_i+\log\mu_i)\right] \\
&= \exp\left[\sum_{i\in I}\log\lambda_i+\sum_{i\in I}\log\mu_i\right] = \exp\left[\sum_{i\in I}\log\lambda_i\right]\,\exp\left[\sum_{i\in I}\log\mu_i\right] \\
&= \det_\zeta A\,\det_\zeta B. \qquad (11)
\end{aligned}$$

But this is *not* true, and *only one* of these steps fails to be true. Below we provide some specific examples to help the reader understand where the problem is.

Actually, very much related with this is the fact that the zeta function trace

$$\mathrm{tr}_\zeta A = \sum_{i\in I}\lambda_i = \zeta_A(s=-1) \qquad (12)$$

fails to satisfy the additive property: in general

$$\mathrm{tr}_\zeta(A+B)\neq \mathrm{tr}_\zeta A+\mathrm{tr}_\zeta B, \qquad (13)$$

for, again, this is a regularized trace (involves analytical continuation) which is used with non trace-class operators (see also[4] for the general definition of the trace).

As an example, consider the following commuting linear operators in an infinite-dimensional space, given in diagonal form by:

$$A_1=\mathrm{diag}\ (1,2,3,4,\ldots), \qquad A_2=\mathrm{diag}\ (1,1,1,1,\ldots), \qquad (14)$$

and their sum

$$A_1+A_2=\mathrm{diag}\ (2,3,4,5,\ldots). \qquad (15)$$

The corresponding ζ-traces are easily obtained:

$$\begin{aligned}
&\mathrm{tr}_\zeta A_1=\zeta_R(-1)=-\frac{1}{12}, \qquad \mathrm{tr}_\zeta A_2=\zeta_R(0)=-\frac{1}{2}, \\
&\mathrm{tr}_\zeta(A_1+A_2)=\zeta_R(-1)-1=-\frac{13}{12}, \qquad (16)
\end{aligned}$$

ζ_R being the Riemann zeta function. The last trace has been calculated according to the rules of infinite series summation (see e.g., Hardy[9]). We observe that

$$\mathrm{tr}_\zeta(A_1 + A_2) - \mathrm{tr}_\zeta A_1 - \mathrm{tr}_\zeta A_2 = -\frac{1}{2} \neq 0. \tag{17}$$

Unlike for ordinary, finite dimensional determinants, for which we have the property: $\det(AB) = \det(A)\det(B)$, for zeta determinants one rather has to consider, in general, an additional piece (called *anomaly* or *defect*). It is usually written as

$$a(A,B) = \ln \frac{\det(AB)}{\det(A)\det(B)} \tag{18}$$

or

$$a\left(A,B\right) = \zeta_A'\left(0\right) + \zeta_B'\left(0\right) - \zeta_{AB}'\left(0\right). \tag{19}$$

Thus the anomaly $a(A,B)$ will vanish if the derivatives at $s = 0$ of the respective zeta function satisfy the additive property. There is an explicit expression, due to Wodzicki, for $a\left(A,B\right)$, that simplifies enormously the calculation of the multiplicative anomaly in many cases.[1]

III. Understanding zeta traces and zeta determinants

There exist many examples of simple cases with and without multiplicative anomaly.[7,8] We give now a condition that guarantees its absence. Consider the two following zeta functions:

$$\zeta_A\left(s\right) = \sum_i \lambda_i^{-s}, \tag{20}$$

$$\zeta_B\left(s\right) = \sum_i \left(c\lambda_i^{\alpha}\right)^{-s} = c^{-s}\zeta_A\left(\alpha s\right), \text{ with } c, \alpha \in \mathbf{R}. \tag{21}$$

The zeta function associated with the product of the eigenvalues is

$$\zeta_{AB}\left(s\right) = \sum_i \left(c\lambda_i^{\alpha+1}\right)^{-s} = c^{-s}\zeta_A\left(\left(\alpha+1\right)s\right), \tag{22}$$

and thus

$$\zeta_{AB}(s) = c^{-s}\zeta_A\left(\left(\alpha+1\right)s\right) \neq \zeta_A\left(s\right) + c^{-s}\zeta_A\left(\alpha s\right) \tag{23}$$

Taking the derivative and performing the substitution $s=0$, we have that:

$$\zeta'_{AB}(0) = -\ln c\,\zeta_A(0) + (\alpha+1)\zeta'_A(0) = \zeta'_A(0) + \zeta'_B(0)\,. \tag{24}$$

Therefore, in spite of the fact that the two zeta functions are different, their respective derivatives at zero are equal. This is enough to guarantee the absence of the multiplicative anomaly, namely $a(A,B)=0$. What is quite a general situation, since we have not fixed the λ_i at all. We have only played with the relative difference between the spectra.

A rather different thing is to consider two spectra which are related by an *additive* constant:

$$\mu_i = \lambda_i + c. \tag{25}$$

For simplicity, let us restrict our analysis to the specific example

$$\lambda_n = n, \qquad \mu_n = n+1, \qquad n = 1,2,3,... \tag{26}$$

Thus

$$\zeta_A(s) = \zeta_R(s), \qquad \zeta_B(s) = \zeta_R(s) - 1, \tag{27}$$

while the zeta function of the product is of Epstein type[7]:

$$\zeta_A(s) = \sum_{n=1}^{\infty}(n^2+n)^{-s} = \sum_{n=1}^{\infty} \frac{\Gamma(n+s)\,2^{-2n}}{n!\,\Gamma(s)}\,\zeta_H(2(n+s),3/2). \tag{28}$$

Thus

$$\zeta'_A(0) + \zeta'_B(0) = 2\zeta'_R(0) = -\ln(2\pi), \tag{29}$$

while

$$\zeta'_{AB}(0) = \sum_{n=1}^{\infty} \frac{2^{-2n}}{n}\,\zeta_H(2n,3/2), \tag{30}$$

which are not equal. Numerically

$$\zeta'_{AB}(0) = 0.4417, \qquad \zeta'_A(0) + \zeta'_B(0) = -1.8379, \tag{31}$$

even the signs are different and the anomaly, in such a simple case, is larger in absolute value than the individual results themselves:

$$a = \zeta'_A(0) + \zeta'_B(0) - \zeta'_{AB}(0) = -2.2796. \tag{32}$$

Up to now, we have addressed and tried to explain the problem by looking carefully in the various zeta functions involved in the process. Nevertheless, we can gain a new insight into the multiplicative anomaly issue through consideration of the factorizability properties of the corresponding zeta functions, an analysis important by itself in, e.g., number theory.

IV. Zeta function factorizations and the multiplicative anomaly

As explained, the main practical consequence about the existence of the multiplicative anomaly is that, if, e.g. we want to compute

$$\det A = \det(BC), \tag{33}$$

from the (in principle simpler) determinants $\det B$ and $\det C$, we have to take also into account $a\,(A,B)$. This is specially important when different factorizations of A, say $A = BC$ and $A = B'C'$, are alternatively considered.[8] We begin by introducing the associated zeta functions that we would use in the computation of the factor determinants:

$$\zeta_B\,(s) = \sum_i \lambda_i^{-s}, \tag{34}$$

$$\zeta_C\,(s) = \sum_j \mu_j^{-s}. \tag{35}$$

But, instead of applying the usual and direct procedure as before, here we shall deal with the product of these two zeta functions:

$$\zeta_D\,(s) = \zeta_B\,(s)\,\zeta_C\,(s) = \sum_i \lambda_i^{-s} \sum_j \mu_j^{-s}. \tag{36}$$

Note that this is the zeta function of an operator, D, which is different from the previous A. Actually, $D = B \otimes C$, as is immediate to realize. In fact, from

$$\zeta_{B\otimes C}(s) = \sum_{i,j} \left(\lambda_i \mu_j\right)^{-s} = \sum_i \lambda_i^{-s} \sum_j \mu_j^{-s}, \quad \text{Re}\ s > \max\,\{\alpha, \beta\}, \tag{37}$$

being α, β, the abscissas of convergence of the individual series, and owing to the uniqueness of the asymptotic continuation to the rest of the complex plane, it turns out that

$$\zeta_{B\otimes C}(s) = \zeta_B(s)\zeta_C(s). \tag{38}$$

In particular,

$$\zeta_{B\otimes C}(s=-1) = \zeta_B(s=-1)\zeta_C(s=-1), \tag{39}$$

that is

$$\mathrm{tr}_\zeta(B \otimes C) = \mathrm{tr}_\zeta B \ \mathrm{tr}_\zeta C, \tag{40}$$

which extends the corresponding property known to hold in finite dimensions.

Now, consider the respective determinants. Recall, to begin with, that in the finite case we have

$$\det(B \otimes C) = (\det B)^{\dim\, C} (\det C)^{\dim\, B}, \tag{41}$$

where the dimensions refer to the spaces where the respective operators act. We will now prove that this equation is maintained in the infinite dimensional situation (we will drop the ζ label from the determinants, from now on). In fact, we have (recall that $\zeta_B(0)$ is the zeta *regularized dimension* of the space in which B acts, and same for the rest):

$$\det B = \exp[-\zeta_B'(0)], \qquad \det C = \exp[-\zeta_C'(0)], \tag{42}$$

$$\det D = \exp[-\zeta_D'(0)] = \exp[-\zeta_B'(0)\zeta_C(0) - \zeta_B(0)\zeta_C'(0)], \tag{43}$$

and we thus see, that

$$\det(D) = (\det B)^{\zeta_C(0)} (\det C)^{\zeta_B(0)}. \tag{44}$$

In the particular case when $\zeta_B(s)$ and $\zeta_C(s)$ have the same value at zero (the two operators act on a space of the same dimension),[10] $\zeta_B(0) = \zeta_C(0) \equiv \widetilde{\zeta}(0)$, we get

$$\det(D) = (\det B \det C)^{\widetilde{\zeta}(0)} \tag{45}$$

We have thus shown that the computation of $\det B \det C$ is, in a way, as close to that of det $(B \otimes C)$ as it is to that of det (BC), provided when both operators act on the same space and can be multiplied. In fact, the determinant of their tensor product is given in terms of the product of the determinants of the individual operators by introducing the regularized dimension of the space where they act. Formally, it is a kind of 'exponential anomaly'. But notice that this is actually no 'anomaly', since the exponent is constant (e.g., it does not depend on the particular operators B and C chosen) and it is always equal to the regularized dimension of the space (as it should!). When $\zeta_B(0) = \zeta_C(0) = \widetilde{\zeta}(0)$, let us compare in more detail the two expressions: the one for the multiplicative anomaly

$$\det(BC) = \det B \det C \, e^{a(B,C)}, \tag{46}$$

with the other for the exponential 'anomaly', thus:

$$\exp(a(B,C)) = \frac{\det(BC)}{\det B \det C} = \frac{\det A}{(\det D)^{\frac{1}{\zeta(0)}}}. \tag{47}$$

This equation seems somewhat artificial, no wonder since it links two non-directly related quantities, as explained above. It can nevertheless be useful in practical determinations of the multiplicative anomaly.

Some consequences and examples. In general, if one is dealing with factorizations of the type:

$$\zeta_A(s) = \prod_i \zeta_{A_i}(s), \tag{48}$$

the determinants are related as $\det A = \prod_i (\det A_i)^{\prod_{j \neq i} \zeta_j(0)}$. This can be useful for the computation of determinants of multidimensional zeta functions, once its factorization is known. For a general $m-$dimensional zeta function, we can write its factorization as: $\zeta(s) = \prod_i^m \zeta_i^{d_i}(s)$ where d_i specifies the dimension of the zeta function, with $m = \sum_i d_i$.

A number of different examples can be worked out. For instance, if the zeta functions factors are zero at the origin, then the associated multidimensional determinant is one. This is what happens, for example, for the product of harmonic oscillators :

$$\prod_{n_1=0}^{\infty} \cdots \prod_{n_k=0}^{\infty} \left(n_1 + \frac{1}{2}\right) \cdots \left(n_k + \frac{1}{2}\right) = 1. \tag{49}$$

Actually, with little more effort a more general case can be considered:

$$\lambda_{n_1 \dots n_k} = (n_1 + c_1) \cdots (n_k + c_k), \qquad n_1, \dots, n_k = 0, 1, 2, 3, \dots \tag{50}$$

Here

$$\zeta(s) = \prod_{j=1}^{k} \zeta_j(s), \qquad \zeta_j(s) \equiv \zeta_H(s, c_j). \tag{51}$$

Recalling that

$$\zeta_H(0, c_j) = \frac{1}{2} - c_j, \qquad \zeta'_H(0, c_j) = \ln \Gamma(c_j) - \frac{1}{2}\ln(2\pi), \tag{52}$$

we get

$$\det A = \prod_{j=1}^{k} \det(A_j)^{\prod_{i\neq j} \zeta_H(0,c_j)} = \prod_{j=1}^{k} \left(\frac{\sqrt{2\pi}}{\Gamma(c_j)} \right)^{\prod_{i\neq j}(1/2-c_i)} , \qquad (53)$$

which reduces to the expression above, Eq. (49), in the particular example considered. This is a nice result of the regularization method.

A second example is the case of a multiple factorization, $\zeta^{(N)}(s) = \prod_{i=1}^{N} \zeta_i(s)$, in which at least one of the zeta functions evaluated at the origin is zero (without loosing generality let us choose $\zeta_1(0) = 0$). Then, the determinant associated with $\zeta^{(N)}(s)$ is just

$$\left(e^{-\zeta_1'(0)} \right)^{\prod_{i=2}^{\infty} \zeta_i(0)} , \qquad (54)$$

that is, the determinant of the zeta function which is zero at the origin, exponentiated with the product of the other zeta functions at zero. Different situations of this type could be discussed.

V. Beyond ΨDOs: the case of complex poles

In this concluding section, we want to comment on the appearance, in some important situations, of complex poles, and on its relationship with the multiplicative anomaly and with factorizations. We begin by paying some attention to the anomaly-free case (20)-(21) studied in Sect. 3. This case corresponds to two commuting operators, for which there is a simple expression for the multiplicative anomaly, due to Wodzicki[1]:

$$a\,(A,B) = \frac{\mathrm{res}\ \left[\left(\ln \left(A^b B^{-a} \right) \right)^2 \right]}{2ab\,(a+b)}, \qquad (55)$$

where $a > 0$ and $b > 0$ are the orders of A and B respectively.

In spite of the generality of (20) and (21), it is clear that this is not the most general case inside the class of commuting operators. To begin with, the fact that one is a function of the other is a sufficient but not a necessary condition for the commutation of the operators (think of the operators involved in the quantum mechanics of the hydrogen atom, for example). In addition, one may also argue that a more general function than $\mu_n = f(\lambda_n) = c\lambda_n^\alpha$ may be considered as well. For example, an exponential function $f(\lambda_n) = \exp(\lambda_n)$. It can be readily seen that with such a choice we are outside the realm of pseudodifferential operators. For instance, just with $\lambda_n = n$, then $\mu_n = \exp(n)$, and then the associated zeta function is a geometric series:

$$\zeta_B(s) = \sum_{n=1}^{\infty} \mathrm{e}^{-ns} = \frac{1}{\mathrm{e}^s - 1}, \tag{56}$$

giving rise to infinitely many complex poles. Nevertheless, this spectrum is indeed physical, as shown in Ref. 11, and related to q-deformations[11] and to fractal geometry[12] as well. Thus, it is also rather reasonable to expect that associated regularized expressions (such as determinants) may be of physical interest as well. In principle, one can proceed identically -depending on the precise meromorphic structure of the corresponding zeta function- with the formal definition. Likewise, note that the case (20) and (21), not only holds for $\alpha \in \mathbf{R}$, but also for $\alpha \in \mathbf{C}$, as can be readily seen from $(22) - (24)$. Therefore, it is still anomaly free but notice that a complex α introduces complex poles (just as a simple example, consider $\mu_n = n$, then the complex α rotates the pole at $s = 1$ to $s = \alpha^{-1}$).

There are other circumstances where we are outside the pseudodifferential operator framework, but there is still interest in the short time asymptotics of the heat kernel or in zeta determinants. This is exactly the case, for instance, when considering heat kernels in non-commutative spaces[13] and when studying products of prime numbers,[14] respectively. Indeed, the zeta function associated to the prime numbers is known,[15]

$$\mathcal{P}(s) = \sum_p p^{-s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns), \tag{57}$$

where p are the prime numbers and $\mu(n)$ is the Möbius function. Note that this function has a rich pattern of logarithmic singularities in the complex plane but still the associated *determinant* is of interest and actually follows directly from the derivative of $\mathcal{P}(s)$.[14] Additionally, in this type of regularized products, one can look at multiplicative anomalies as well. For example, following,[14] one can consider the Euler product representation of Riemann's zeta function:

$$\zeta(s) = \frac{\prod_p p^s}{\prod_p p^s - 1}, \quad \mathrm{Re}\ s > 1. \tag{58}$$

This expression, considered together with $\mathcal{P}(s)$, gives rise to the following result[14]:

$$\prod_p (p-1) = 0 \quad \text{and} \quad \prod_p \left(p^2 - 1\right) = 48\pi^2, \tag{59}$$

and the appearance of a multiplicative anomaly is manifest.

Acknowledgments

EE is indebted to the Mathematics Department, MIT, especially Dan Freedman, for the warm hospitality. One of the author's (E.E.) investigation has been partially supported by DGI/SGPI (Spain), project BFM2000-0810, and by CIRIT (Generalitat de Catalunya), contract 2001SGR-00427. MT is indebted to Michel Lapidus for many useful and interesting discussions on zeta functions, and also for warm hospitality at University of California at Riverside.

Appendix A. Remarks on zeta factorizations.

We have seen how the discussion of the multiplicative anomaly of determinants, has lead us, in a natural way, to the construction of a zeta function from the product of other zeta functions. Generically, and following the previous notation, let us envisage

$$\zeta_D(s) = \zeta_B(s)\,\zeta_C(s) = \sum_i \lambda_i^{-s} \sum_j \mu_j^{-s}. \tag{A1}$$

This turns out to be an important construction in number theory. Actually, even with the simplest zeta functions as factors, important and sophisticate $\zeta_D(s)$ are obtained. For example, with the Riemann zeta function itself. In fact, from the Euler product of the Riemann zeta function, we know that it has local factors of degree 1 at each prime, while automorphic L functions have local factors of degree 2 at almost all places.[16] This suggests that we can denote such product as

$$L(s) = \zeta_R(s)\,\zeta_R(s-k+1)\,, \tag{A2}$$

with $k \geq 2$. In[16] it is shown that the L function is actually

$$L(s) = \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, n^{-s}, \tag{A3}$$

where σ_k is the arithmetic function (the generalized divisor function, or sum over all the divisor of n to some power), given by

$$\sigma_k(n) = \sum_{d|n} d^k. \tag{A4}$$

This appears naturally in the Chowla-Selberg formula and its generalizations.[7]

This shows, in close relationship with the preceding section, how the product of even the simplest of the zeta functions lead to an interesting object by the process considered above, often with important arithmetic properties (and some of these L functions are useful in analytical approaches to the study of algorithms[20,21]). Even more, in general, increasingly complex L functions are very often constructed or represented by a generic product of simpler L functions.[16]

Nevertheless, it seems apparent that instead of exploiting the useful idea of constructing zeta functions, it may also be worth to look at this relation to the other side, that is, as a decomposition of the zeta function on the l.h.s. into several factors. To illustrate the approach for zeta functions, let us just take into account the two simple examples considered in detail in.[17] The zeta function

$$\zeta(s) = \sum_{m,n\in\mathbf{Z}^2}{}' (m^2+n^2)^{-s}, \tag{A5}$$

with the summation extended over all pairs $(m,n) \neq (0,0)$ in $\mathbf{Z}^2$, can be expressed as:

$$\zeta(s) = 4\zeta_R(s)\cdot L(\chi_4, s). \tag{A6}$$

where $\zeta_R(s)$ is the Riemann zeta function and $L(\chi_4, s)$ is the Dirichlet zeta function corresponding to the character χ_4. Another interesting factorization is the following one, for a different particular case of the two-dimensional Epstein zeta function

$$\zeta(s) = \sum_{m,n\in\mathbf{Z}^2}{}' (m^2+mn+n^2)^{-s} = 6\zeta_R(s)\cdot L(\chi_3, s) \tag{A7}$$

Once again, we see the natural appearance of L functions, whose determinants are of much interest as well (mainly in a number theoretical context; see[18] for a review).

These factorizations are particular cases of a more general situation coming from algebraic considerations in number theory.[16] Very general statements are not always possible, but let us compare the previous with the classical results (due to Dirichlet) concerning primitive quadratic forms of any determinant:

$$Q\left(x\right) = ax_1^2 + bx_1x_2 + cx_2^2, \qquad (a,b,c) = 1 \tag{A8}$$

(the parenthesis meaning here 'maximum common divisor'), with $D = -\det Q = b^2 - 4ac < 0$ (the discriminant of Q), and

$$\chi_D\left(d\right) = \left(\frac{D}{d}\right). \tag{A9}$$

Then, for $n > 0$, $(n, D) = 1$, the character sum:

$$r\left(n; D\right) = \omega_D \sum_{d|n} \chi_D\left(d\right) \tag{A10}$$

gives the number of all representations of n by representatives of forms of all classes of discriminant D. Here ω_D stands for the number of automorphs:

$$\omega_D = \begin{cases} 6, & \text{if } D = -3, \\ 4, & \text{if } D = -4, \\ 2, & \text{if } D < -4. \end{cases} \tag{A11}$$

Notice how the discriminant gives the right character for the L function and the number of automorphs the right prefactor in the previous example of factorization. Nevertheless, we must point out that these previous examples and the posterior discussion looks so simple, due to the fact that the examples correspond to discriminants D for which the class number $h\left(D\right)$ (the number of equivalence classes of primitive binary quadratic forms) is one.

1. Factorization at the level of the heat kernels.

Now, we pay attention to the meaning of the zeta factorization at the level of the respective associated heat kernels. Since A^{-s} and $\exp\left(-tA\right)$ are related by the following expression:

$$A^{-s} = \frac{1}{\Gamma\left(s\right)} \int_0^\infty t^{s-1} \exp\left(-tA\right) dt, \tag{A12}$$

then the zeta function is, up to a gamma function, the Mellin transform of the heat kernel. The interest of this expression, considered together with the factorization property, is that it allows, in a probabilistic context, the product of random variables to be directly performed in Mellin space (in contrast to the better known case of the addition of variables, where Fourier transform is used).[19] Therefore, the zeta factorization implies also a product for the respective heat kernels, but a product in the sense of probability theory, that is, the heat kernel (or in a number theoretical context, the theta function) denotes the probability distribution function of a random variable X_i, and then we have the product $X = \prod_{i=1}^n X_i$. Nevertheless, for this to be exactly correct we should take into account the gamma function for each factor and for the resulting zeta function. For example, in the case of a zeta function with two factors:

$$\begin{aligned} \zeta\left(s\right) = \zeta_1\left(s\right)\zeta_2\left(s\right) \quad &\longrightarrow \quad \Gamma\left(s\right)\Gamma\left(s\right)\zeta\left(s\right) = \Gamma\left(s\right)\zeta_1\left(s\right)\Gamma\left(s\right)\zeta_2\left(s\right) \\ &\longrightarrow \quad K\left(t\right)\cdot\exp\left(-t\right) = K_1\left(t\right)\cdot K_2\left(t\right), \end{aligned} \tag{A13}$$

where, in the last expressions, the products are in the sense explained above, and we have used the fact that $\Gamma(s) = \int_0^\infty t^{s-1} \exp(-t)\, dt$. Thus, the necessary introduction of gamma factors implies that we have to take into account possible products of the main heat kernel with an exponential distribution.

This stochastic point of view seems both interesting from the mathematical side, where a probabilistic interpretation of zeta and theta functions is of interest,[?] and also from a physical perspective, where products of random variables very often constitute a role model of what is known with the name of multiplicative or cascade processes.[23]

Last but not least, the factorization is potentially interesting from the practical point of view in the asymptotic study of the trace of the heat kernel (9) and (10). The contributions can be considered separately, with the exception of the possible coincidence of poles or poles and zeros. This fact introduces interesting phenomena that can be seen with the following example. Consider the product of two Riemann zeta functions,

$$\zeta(s) = \zeta_R(s)\, \zeta_R(s), \tag{A14}$$

which yield the well-known L-function:

$$\zeta(s) = L(s) = \sum_{k=1}^{\infty} \frac{d(k)}{k^s}, \tag{A15}$$

with $d(k)$ the divisor function again. Note the consistency with the previous case (A2)-(A4). The idea is now to construct another zeta function from two very similar factors:

$$\zeta_\varepsilon(s) = \zeta_R(s(1+\varepsilon))\, \zeta_R(s(1-\varepsilon)), \tag{A16}$$

with $\varepsilon > 0$ a very small, real positive number. It seems that these two zeta functions should be almost identical in the whole complex plane, except for the fact that, in the first one, we have a double pole at $s = 1$, while the second has two simple poles at $s = (1+\varepsilon)^{-1}$ and $s = (1-\varepsilon)^{-1}$, very close one from the other for ε small. Note that the point-like structure of a pole allows to play that game. Now, from (9) and (10), it is clear how different the $t \to 0$ expansion of the associated trace of the heat kernel is, in the two cases. In the first case, we have

$$\mathrm{tr}\; e^{-tA_1} \sim -\frac{\log t}{t}, \;\; t \downarrow 0, \tag{A17}$$

in sharp contrast with the second case, where

$$\mathrm{tr}\ e^{-tA_2} \sim \Gamma\left(\frac{1}{1+\varepsilon}\right) t^{-\left(\frac{1}{1+\varepsilon}\right)} + \Gamma\left(\frac{1}{1-\varepsilon}\right) t^{-\left(\frac{1}{1-\varepsilon}\right)},\ t \downarrow 0. \tag{A18}$$

We see that the case where the poles collide possesses a partition function which is much larger the smaller the value of t is (the classical limit). Therefore, the associated partition functions differ considerably in the classical limit. A deeper physical understanding of this phenomena seems to be an interesting open question.

References

[1]M. Wodzicki. *Non-commutative Residue Chapter I.* In *Lecture Notes in Mathematics.* Yu.I. Manin, editor, Vol. 1289, 320 (Springer-Verlag, Berlin, 1987).

[2]K. Kirsten, *Spectral functions in mathematics and physics* (Chapman & Hall/CRC Press, Boca Raton, FL, 2001); D. V. Vassilevich, arXiv:hep-th/0306138.

[3]E. Elizalde, *Ten physical applications of spectral zeta functions* (Springer, Berlin, 1995); E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, *Zeta regularization techniques with applications* (World Sci., Singapore, 1994). A.A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini, Phys. Reports **266** (1996) 1; A.A. Bytsenko, G. Cognola, E. Elizalde, V. Moretti and S. Zerbini, *Analytic aspects of quantum fields* (World Scientific, Singapore, 2003).

[4]V. Mathai, R.B. Melrose and I.M. Singer, math.DG/0206002.

[5]D.B. Ray, Adv. in Math. **4**, 109 (1970); D.B. Ray and I.M. Singer, Adv. in Math. **7**, 145 (1971); Ann. Math. **98**, 154 (1973).

[6]C. Kassel, Seminaire Bourbaki, Vol 1998/89. Asterisque No. 177-178 (1989), Exp. No. 708, 199-229; M. Kontsevich and S. Vishik, in *Functional analysis on the eve of the 21st century.* Edited by S. Gindikin, J. Lepowsky and R.L. Wilson, Prog. Math. (Birkhauser) **131,** 173 (1995)**;** K. Okikiolu, Duke Math. J. **79**, 687 (1995)

[7]E. Elizalde, JHEP **9907,** 015 (1999); J. Comput. Appl. Math. **118,** 125 (2000); Commun. Math. Phys. **198**, 83 (1998).

[8]E. Elizalde, L. Vanzo and S. Zerbini, Commun. Math. Phys. **194,** 613 (1998); E. Elizalde, G. Cognola and S. Zerbini, Nucl. Phys. B **532**, 407 (1998); E. Elizalde, A. Filippi, L. Vanzo, S. Zerbini, Phys. Rev. **D57**, 7430 (1998).

[9]G.H. Hardy, *Divergent Series* (Oxford University Press, Oxford, 1949).

[10]Notice that this is an inescapable requirement if the ordinary product of the two operators is to make any sense.

[11]M. Tierz, arXiv:hep-th/0308121.

[12]M.L. Lapidus and M van Frankenhuysen, *Fractal Geometry and Number Theory (Complex dimensions of fractal strings and zeros of zeta functions).* Research monograph, Birkhäuser, Boston. 2000.

[13]D. V. Vassilevich, arXiv:hep-th/0310144.

[14]E. Muñoz Garcia and R. Perez-Marco, IHES, Report No., IHES/M/03/34. http://www.ihes.fr/PREPRINTS/M03/Resu/resu-M03-34.html

[15]E. Landau and A. Walfisz, Rend. Circ. Mat. Palermo **44,** 82 (1920).

[16]H. Iwaniec, *Topics in Classical Automorphic Forms* (Graduate Studies un Mathematics. Vol 17. American Mathematical Society. 1997).

[17]P. Cartier in *From Number Theory to Physics.* Edited by M. Waldschmidt, P. Moussa, J. M. Luck and C. Itzykson (Springer-Verlag, Berlin, 1992).

[18]H. M. Stark in *From Number Theory to Physics.* Edited by M. Waldschmidt, P. Moussa, J. M. Luck and C. Itzykson (Springer-Verlag, Berlin, 1992).

[19]M. D. Springer, *The algebra of random variables*, Wiley Series in Probability and Mathematical Statistics. 1979.

[20]R. Sedgewick and P. Flajolet, *An introduction to the Analysis of Algorithms*, Addison-Wesley, 1996.

[21]P. Flajolet et al, Theoretical Computer Science **123**, 291 (1994).

[22]P. Biane, J. Pitman and M. Yor, Bull. Amer Math. Soc. **38**, 435 (2001).

[23]D. Sornette, Phys. Rev. E **57**, 4811 (1998).