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SOFT MATRIX MODELS AND CHERN–SIMONS PARTITION FUNCTIONS

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We study the properties of matrix models with soft confining potentials. Their precise mathematical characterization is that their weight function is not determined by its moments. We mainly rely on simple considerations based on orthogonal polynomials and the moment problem. In addition, some of these models are equivalent, by a simple mapping, to matrix models that appear in Chern–Simons theory. The models can be solved with q deformed orthogonal polynomials (Stieltjes–Wigert polynomials), and the deformation parameter turns out to be the usual q parameter in Chern–Simons theory. In this way, we give a matrix model computation of the Chern–Simons partition function on S^3 and show that there are infinitely many matrix models with this partition function.

Keywords: Chern–Simons theory; matrix models; q-orthogonal polynomials; moment problem.

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1. Introduction

Random matrix models[5,17] (for a review see Ref. 8) have attracted great amount of interest both due to its mathematical structure and for its manifold physical applications as well. When studying matrix models, one is especially interested in the solution of matrix models in the large-N approximation, which corresponds to the planar Feynman diagrams.[5,8] These models are characterized by the $(N \times N)$-matrix variable M and by a Hamiltonian $H = \operatorname{Tr} V(M)$. After diagonalization of the matrix, one is able to work in the eigenvalue space, and to consider the following expression for the partition function:

$$Z_M = \int \prod_{i=1}^{\lambda} d\lambda_i \exp(-V(\lambda_i)) \prod_{i<j} (\lambda_i - \lambda_j)^{\beta} \,, \tag{1.1}$$

with $\beta = 1, 2$ or 4, depending on the symmetry. Equivalently, from the perspective of random matrix theory one studies the joint probability distribution function for the N eigenvalues of the matrix.[17] It has the well-known general form:

$$P(x_1, \ldots, x_N) = C_N \prod_{i<j} |x_i - x_j|^\beta \prod_{i=1}^{N} \omega(x_i)$$
$$= C_N \prod_{i<j} |x_i - x_j|^\beta \exp\left[-\sum_{i=1}^{N} V(x_i)\right], \quad (1.2)$$

$\omega(x)$ and $V(x)$ are named weight function and confining potential, respectively. If the elements of the random matrix are believed to be statistically independent from each other, one obtains the quadratic confinement potential $V(x) = x^2$, leading to the Gaussian ensembles of random matrices.[17] In the usual physical applications of matrix models,[8] the confining potentials $V(x)$ are such that the weight function $\omega(x) = \exp[-V(x)]$ is determined by the knowledge of all of its positive integer moments $\{\gamma_n\}_{n=0}^{\infty}$, where $\gamma_n \equiv \int x^n \omega(x) dx$. In this paper, we want to study some general properties of matrix models with a weight function that does not satisfy this property. Interestingly enough, a very particular model in this category shows considerable interest in the context of Chern–Simons theory[13,22] and topological strings.[11,15,23]

We briefly remind now the formalism of orthogonal polynomials. Recall that the relevant quantities, such as the density of states and the correlation functions, are obtained from the two-point kernel[17]:

$$K_N(x, y) = \sum_{k=0}^{N-1} \varphi_k(x)\varphi_k(y), \quad (1.3)$$

with $\varphi_k(x) = \omega(x)p_k(x)$, where $p_k(x)$ are the orthogonal polynomials associated to the weight function $\omega(x)$. This is the fundamental object, due to the following classical result[17]:

$$R_k(x_1, x_2, \ldots, x_k) = \det[K_N(x_p, x_q)]_{q,p=1,2,\ldots,n}, \quad (1.4)$$

where the correlation functions are defined as:

$$R_k(x_1, x_2, \ldots, x_k) \equiv \frac{N!}{(N-k)!} \int_{-\infty}^{\infty} dx_{k+1} \cdots \int_{-\infty}^{\infty} dx_N \, P(x_1, x_2, \ldots, x_N). \quad (1.5)$$

For Hermitian ensembles ($\beta = 2$), the density of states is the kernel at the origin:

$$\rho(x) = \sum_{k=0}^{N-1} \varphi_k(x)^2$$
$$= \omega(x)(p_N(x)p'_{N-1}(x) - p'_N(x)p_{N-1}(x)), \quad (1.6)$$

the second expression follows when applying the Christoffel–Darboux formula.[17,20] For Gaussian ensembles for example, this quantity tends to the well-known semi-circle law in the limit $N \to \infty$.[17] In this work we shall consider Hermitian ensembles, and this expression for the density of states will be useful to understand relevant conceptual points in our discussion.

As mentioned, we are concerned with the special properties of matrix models with an indeterminate weight function. Our discussion is mainly based on the classical moment problem[2] (see Ref. 18 for a recent review, that we will follow). The connection with random matrix theory can be readily guessed by the fact that only the moments play a role in the orthogonalization procedure. Therefore the two sets, the moments and the orthogonal polynomials, are essentially equivalent.

The paper is organized as follows: In the next section, we give a short introduction to Chern–Simons theory and, focussing on these recent developments, we show how the matrix models in Refs. 1 and 14 are directly related with a Hermitian matrix model with a log-normal weight function. In this way, one can do exact computations through the associated orthogonal polynomials. To illustrate this, we compute the partition function of Chern–Simons theory on S^3 with gauge group $\mathrm{U}(N)$. Interestingly enough, the use of q-deformed orthogonal polynomials readily leads to the natural parameter of Chern–Simons theory $q = e^{-g_s} = e^{\frac{2\pi i}{N+k}}$. Therefore, the orthogonal polynomials approach is a method for nonperturbative solutions in Chern–Simons theory. In Sec. 3, we show that there are actually infinitely many matrix models with the same partition function. Finally, we show that this is a general feature of models with very weak confining potential. To conclude, we present a summary and possible directions for further research.

2. Chern–Simons Theory on S^3: Matrix Model Formulation

We begin by outlining basic and well-known facts about Chern–Simons theory, emphasizing recent results and trends that connect Chern–Simons theory with matrix models.[1,14] As it is well known, Chern–Simons theory is a topological quantum field theory whose action is built out of a Chern–Simons term involving as gauge field a gauge connection associated to a group G on a three-manifold M. The action is:

$$S(A) = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \tag{2.1}$$

with k an integer. The natural associated observables are the correlators of Wilson loops, and its main interest come from the fact that these correlators lead to quantum-group polynomial invariants of knots and links.[22] For a review of the field since the seminal work,[22] see Ref. 13.

We mention now some few developments related to Chern–Simons theory. As mentioned in Ref. 23, Chern–Simons gauge theory has a string description in the sense of 't Hooft.[12] Indeed, as it is well known, gauge theories with the $\mathrm{SU}(N)$ group admit a large-N expansion. In these expansions, correlators are expanded in powers of $1/N$ while keeping the 't Hooft coupling fixed $t = xN$, with x the coupling constant of the gauge theory. In the case of Chern–Simons theory, this large-N expansion is reminiscent of a string theory expansion. This connection between Chern–Simons and topological strings was first pointed out by Witten[23] and has been extended in Ref. 11. For example, in Ref. 23 it is shown that if one wraps N D-branes on M in T^*M, then the associated topological A-model is a $\mathrm{U}(N)$ Chern–Simons theory on the three-manifold M.

On the other hand, regarding matrix models, in Ref. 10 it was already pointed out that the structure of the partition function of Chern–Simons theory on S^3 with gauge group $\mathrm{SU}(N)$, resembles the usual expression for the partition function of a one matrix model in terms of its associated orthogonal polynomials. Moreover, in Ref. 14, it is shown that the partition function of Chern–Simons theory on S^3 and with gauge group $\mathrm{U}(N)$, is given by:

$$Z = \frac{e^{-\frac{g_s}{12}N(N^2-1)}}{N!} \int \prod_{i<j} \left(2\sinh\frac{u_i - u_j}{2}\right)^2 \prod_{i=1}^{N} e^{-u_i^2/2g_s} \frac{du_i}{2\pi}\,, \tag{2.2}$$

that describes open topological A strings on T^*S^3 with N branes wrapping S^3 (see the details in Ref. 14). These types of models have been further considered in Ref. 1. In Ref. 7, it was shown that topological strings for B-branes are equivalent to Hermitian matrix models, then the idea in Ref. 1 is to obtain the results in Ref. 14, by applying mirror symmetry to obtain B-brane matrix models. We will comment further on the results in Ref. 1 later on. Now, we mainly focus on the study of (2.2). For recent reviews along these lines see Ref. 15.

We remind that, as usual in Chern–Simons theory, the string coupling constant g_s is related with the k in (2.1) by:

$$g_s = \frac{2\pi i}{k+N}\,. \tag{2.3}$$

As explained in Ref. 14, the limit of the parameter $g_s \to 0$, leads to the usual Gaussian unitary ensemble. One can argue, for example, that in this limit, the Gaussian becomes a Dirac delta function, and therefore the results are independent of the correlation factor, or level repulsion. In Refs. 1 and 14, this model is essentially studied through the consideration of averages (in a Gaussian unitary ensemble) of the following quantities:

$$e^{-\sum_{k=1}^{\infty} a_k \sigma_k(u)}\,, \quad \text{with} \quad a_k = \frac{B_{2k}}{k(2k)!} \quad \text{and} \quad \sigma_k(u) = \sum_{i<j}(u_i - u_j)^{2k}\,. \tag{2.4}$$

This method leads to a great amount of perturbative information. Nevertheless, we note that the connection with the usual Hermitian matrix models, i.e. of the type given by (1.1), is much simpler. Let us consider the following simple change of variables:

$$e^{u_i} = x_i\,, \tag{2.5}$$

then, (2.2) reads:

$$Z = \frac{e^{-\frac{g_s}{12}N(N^2-1)}}{N!} \int \prod_{i<j}(x_i - x_j)^2 \prod_{i=1}^{N} x_i^{-N} e^{-\log^2 x_i/2g_s} \frac{dx_i}{2\pi} . \tag{2.6}$$

The factor $\prod_i x_i^{-(N-1)}$ can be readily absorbed by considering a simple but remarkable property of the log-normal function $\omega(x) = e^{-\log^2 x_i/2g_s}$:

$$\omega(xq) = \sqrt{q}x\omega(x) . \tag{2.7}$$

Note that this functional equation is just a particular case of the following elementary identity:

$$\begin{aligned} e^{-k^2 \log^2(e^{(-\alpha/2k^2)}x)} &= e^{-k^2(\log^2 x + \alpha^2/4k^4 - \frac{\alpha}{k^2}\log x)} \\ &= e^{-\alpha^2/4k^2} x^{\alpha} e^{-k^2 \log^2 x} . \end{aligned} \tag{2.8}$$

Therefore, the Chern–Simons matrix model is directly related to an ordinary Hermitian ensemble with log-normal weight function $\omega(x) = e^{-\log^2 x_i/2g_s}$. Equivalently, one could have used the change of variables $\exp(u_i + Ng_s) = x_i$. Notice that the term Ng_s is necessary to cancel, together with the term coming from the Jacobian, the contribution $\prod_i x_i^{-(N-1)} = \prod_{i<j} \frac{1}{x_i x_j}$. Actually, this method was already employed in Ref. 9, where the same matrix model seems to appear in a very different context. One of our goals is to make contact between this model and Chern–Simons theory. As we shall see, this will come out rather easily, due to the existence of a closed system of orthogonal polynomials associated to the log-normal weight. These polynomials are q-deformed polynomials known as Stieltjes–Wigert polynomials.[20] This opens the possibility of a complete solution for the model (2.2) of Refs. 1 and 14 (see also Ref. 9), using the classical results for Hermitian ensembles and these polynomials. Secondly, we shall show, giving explicit examples, that there exist infinitely many matrix models — deformations of (2.6) — with exactly the same (Chern–Simons) partition function.

2.1. *Chern–Simons partition function through the matrix model computation*

Using the previous results and the Stieltjes–Wigert orthogonal polynomials,[20] we can find a complete solution of the matrix model considered in Ref. 14, and, in particular, for the partition function. For this, we can use the following well-known result for the partition function in terms of the orthogonal polynomials[8]:

$$\begin{aligned} Z &= \int \cdots \int \prod_{i=1}^{N} \omega(x_i) dx_i \prod_{i<l}(x_i - x_l)^2 \\ &= \frac{N!}{\prod_{i=0}^{N-1} a_i^2} = N! a_0^{-2N} \prod_{i=1}^{N-1} \left(\left(\frac{a_{i-1}}{a_i} \right)^2 \right)^{N-i} , \end{aligned} \tag{2.9}$$

where the coefficient a_i is:

$$p_i(x) = a_i x^i + \cdots \,. \tag{2.10}$$

The first step is to compute the Z associated to the Stieltjes–Wigert orthogonal polynomials. The coefficients are given by[20]:

$$a_j = q^{(j+1/2)^2}\{(1-q)\cdots(1-q^j)\}^{-1/2}\,, \tag{2.11}$$

then

$$\left(\frac{a_{j-1}}{a_j}\right)^2 = q^{-4j}(1-q^j) \tag{2.12}$$

and $a_0 = q^{1/4}$. Therefore:

$$\begin{aligned} Z_{sw} &= N! q^{-N/2} \prod_{j=1}^{N-1} q^{-4j(N-j)}(1-q^j)^{N-j} \\ &= N! q^{-\frac{1}{6}N(2N-1)(2N+1)} \prod_{j=1}^{N-1} (1-q^j)^{N-j}\,, \end{aligned} \tag{2.13}$$

but we are interested in:

$$\begin{aligned} Z_{\sinh} &= \int \prod_{i=1}^{N} \frac{du_i}{2\pi} e^{-\frac{u_i^2}{2g_s}} \prod_{i<j}\left(2\sinh\left(\frac{u_i-u_j}{2}\right)\right)^2 \\ &= (2\pi)^{-N} e^{\frac{-N^3 g_s}{2}} \int \prod_{i=1}^{N} dx_i\, e^{-\frac{\log^2(x_i)}{2g_s}} \prod_{i<j}(x_i-x_j)^2\,. \end{aligned} \tag{2.14}$$

Making the identification $k^2 = \frac{1}{2g_s}$, and therefore $q = e^{-g_s}$. It is remarkable that we have naturally obtained, from this q deformed orthogonal polynomials, the usual q parameter in Chern–Simons theory. The partition function with the usual Stieltjes–Wigert weight, corresponds to:

$$\begin{aligned} Z_{sw} &= \int \prod_{i=1}^{N} dx_i\, \pi^{-1/2} k e^{-k^2 \log^2(x_i)} \prod_{i<j}(x_i-x_j)^2 \\ &= (2\pi g_s)^{-N/2} \int \prod_{i=1}^{N} e^{-\frac{\log^2(x_i)}{2g_s}} dx_i \prod_{i<j}(x_i-x_j)^2\,. \end{aligned} \tag{2.15}$$

Therefore:

$$Z_{\sinh} = \left(\frac{g_s}{2\pi}\right)^{N/2} N! e^{\frac{1}{6}g_s N(N^2-1)} \prod_{j=1}^{N-1}(1-q^j)^{N-j}\,. \tag{2.16}$$

To make explicit contact with the typical expressions for the Chern–Simons partition functions (as in Ref. 10 for example) we make some transformations on the product term.

$$\prod_{j=1}^{N-1}(1-q^j)^{N-j} = \prod_{j=1}^{N-1}\left(2\sinh\frac{g_s j}{2}\right)^{N-j} e^{-\frac{g_s j(N-j)}{2}}$$

$$= e^{\frac{-1}{12}g_s N(N^2-1)} e^{\frac{i\pi}{4}N(N-1)} \cdot \prod_{j=1}^{N-1}\left(2\sin\frac{g_s j}{2i}\right)^{N-j} . \quad (2.17)$$

The final expression for the partition function is[14]:

$$Z = \frac{e^{-\frac{1}{12}g_s N(N^2-1)}}{N!} Z_{\sinh}$$

$$= \left(\frac{g_s}{2\pi}\right)^{N/2} e^{\frac{i\pi}{4}N(N-1)} \prod_{j=1}^{N-1}\left(2\sin\frac{g_s j}{2i}\right)^{N-j} , \quad (2.18)$$

and since $g_s = \frac{2\pi i}{k+N}$, we finally find:

$$Z = e^{\frac{1}{4}i\pi N^2}(k+N)^{-N/2} \prod_{j=1}^{N-1}\left(2\sin\frac{\pi j}{k+N}\right)^{N-j} . \quad (2.19)$$

It was already mentioned in Ref. 10, that the structure of the partition function resembled very much the general expression for the partition function of a one-matrix model in terms of the associated orthogonal polynomials. Here, based on the result in Ref. 14, we are making this statement precise through the explicit computation. Notice that the differences between the SU(N) and U(N) comes essentially from the contribution of the partition function of U(1) (see, for example, the first reference in Ref. 15 for more details).

3. Infinitely Many Matrix Models with the Same Chern–Simons Partition Function

Here we shall show that the matrix models involved that appear in Chern–Simons theory present a novelty in comparison with most of known matrix models (like the ones that are relevant in 2D quantum gravity). Broadly speaking, its weight function is so broad that there is a non-uniqueness issue that leads to very practical and concrete results for the matrix model: there are infinitely many models with identical properties. In particular, identical (Chern–Simons) partition function. We begin by briefly reviewing some introductory results in the theory of the moment problem.

3.1. *Relevant results from the moment problem*

The two basic moment problems are[18]:

3.1.1. *Hamburger moment problem*

Given a sequence of reals $\gamma_0, \gamma_1, \ldots$, when is there a measure, $d\rho$, on $(-\infty, \infty)$ so that

$$\gamma_n = \int_{-\infty}^{\infty} x^n \, d\rho(x) \tag{3.1}$$

and if such measure exists, is it unique?

3.1.2. *Stieltjes moment problem*

Given a sequence of reals $\gamma_0, \gamma_1, \ldots$, when is there a measure, $d\rho$, on $(0, \infty)$ so that

$$\gamma_n = \int_{0}^{\infty} x^n \, d\rho(x) \tag{3.2}$$

and if such measure exists, is it unique?

In our case, we are not concerned with the question regarding existence, since we always refer to the weight function $\omega(x)$ and its associated moments. Then, the main point is to know whether we are in the determinate or indeterminate case. To fix ideas, let us mention that there is a natural boundary for functions of the form $e^{-V(x)}$, with $V(x)$ of polynomial form. In the Hamburger case, the boundary is given by $V(x) = |x|^{\alpha}$ with $\alpha = 1$, while in the Stieltjes case it is given by $V(x) = |x|^{\alpha}$ with $\alpha = 1/2$.

Then, in the usual matrix models considered (as reviewed in Ref. 8 for example), one is essentially always in the determined case.

Consider now the following generalization of the log-normal weight:

$$f_{\theta}(x) = e^{-\log^2 x}(1 + \vartheta \sin(2\pi \log x)) \,, \quad \text{with} \quad \vartheta \in [-1, 1] \,. \tag{3.3}$$

The moments of this function are[19]:

$$\gamma_n = \int_{0}^{\infty} f_{\theta}(x) x^n \, dx = \sqrt{\pi} e^{\frac{1}{4}(n^2+1)} \,. \tag{3.4}$$

Note that all the integer moments are completely independent of the parameter ϑ. This means that all the functions in the family $f_{\theta}(x)$ have the same moments. Thus, they are all undetermined by them. Conversely, one can say that the set of moments $\gamma_n = \sqrt{\pi} e^{\frac{1}{4}(n^2+1)}$, is an indeterminate set. We present now a detailed consideration of this family of functions. The main point is the computation of its Mellin transform.

First, note that for $\theta = 0$ we have the log-normal distribution $e^{-\log^2(x)}$. Its Mellin transform is:

$$M[f(x); s] = e^{\pi^2/4} \operatorname{Erf}\left(-\frac{s}{2}\right) . \tag{3.5}$$

The other part of the Stieltjes function is $e^{-\log^2 x} \sin(2\pi \ln x)$. From (3.4), we know that all of its integer moments are zero. The most enlightening possibility is to compute its Mellin transform:

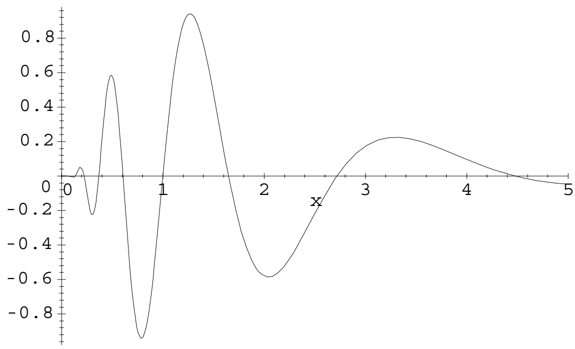

Fig. 1. The oscillatory term in $f_\theta(x)$.

$$I \equiv \int_0^\infty x^{s-1} e^{-\log^2(x)} \sin(2\pi \log x) dx\,. \tag{3.6}$$

We use the change of variables $y = -\frac{s+1}{2} + \log x$. The integral becomes:

$$\begin{aligned} I &= \int_{-\infty}^{\infty} e^{-y^2+\frac{1}{4}(s^2+1)} \sin\left[2\pi\left(y + \frac{s+1}{2}\right)\right] dy \\ &= \sqrt{\pi} e^{\frac{1}{4}(s^2+1)} \sin(\pi(1+s))\,. \end{aligned} \tag{3.7}$$

The change of variables is the one by Stieltjes,[19] with a generic, complex s instead of an integer k. Clearly, after the change of variables, since sin is a periodic and odd function, its value is zero at $s = k$. Nevertheless, the result is even clearer when looking at the full Mellin transform: the contribution of the additional function is zero only for integer values of s.

As a consequence, for example, we have that any of the following functions have the same integer moments.

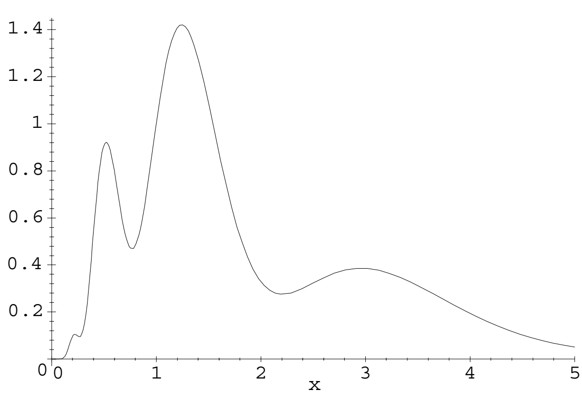

Fig. 2. $\theta = 1/2$.

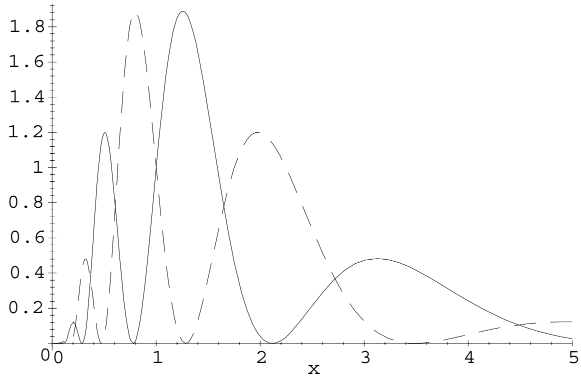

Fig. 3. $\theta = 1$ (solid) and $\theta = -1$ (dashed).

Consequently, it is manifest now that there are infinitely many matrix models with exactly the same partition function. Therefore, it turns out that the partition function of U(N) Chern–Simons theory on S^3 can be given by infinitely many different matrix models. From what we have seen, one possibility is:

$$Z_N = \int \prod_{i=1}^{N} e^{-u_i^2/2g_s}(1 + \theta \sin(2\pi(u_i/g_s + N)))\frac{du_i}{2\pi} \prod_{i<j}\left(2\sinh\frac{u_i - u_j}{2}\right)^2 , \quad (3.8)$$

as always, with $\theta \in [-1, 1]$. Notice that due to the correspondence between (2.2) and (2.6) through (2.5), the density of states of (3.8) does *not* satisfy the property:

$$\int_{-\infty}^{\infty} \rho_\theta(u) u^n \, du = \gamma_n \,, \quad (3.9)$$

(i.e. the moments are different for each matrix model). But rather:

$$\int_{-\infty}^{\infty} \rho_\theta(e^u) e^{(n+1)(u+Ng_s)} \, du = \tilde{\gamma}_n \,, \quad \text{with} \quad n \in Z \,. \quad (3.10)$$

To extract further physical consequences of these results and its implications for the connection between matrix models and topological quantum field theories seems an interesting open question.

4. On Matrix Models with Soft Potentials

In this concluding section we show that the properties exhibited by the Chern–Simons matrix model are typical of models with weakly confining (soft) potentials. The mathematically precise description is that the potential is such that $e^{-V(x)}$ is a function that is not determined by its moments. This feature has already been discussed in the context of random matrix theory and also in the particular case of q-deformed matrix models.[16] In this last case, the one we are mainly interested, the non-uniqueness is strong since the potentials are asymptotically $V(x) \sim \log^2 x$

for[16] $x \to \infty$ and we have already mentioned that we are in the soft regime when the confinement provided by the potential is weaker than $V(x) \sim x$.

Employing basic results on indeterminate functions $w(x)$, and the classical random matrix formulas, one can prove that the density of states of a matrix model is an indeterminate function if its weight function is indeterminate by its moments. This can be readily shown if we consider the expression for the density of states (for any N) (1.6) and Krein proposition,[18] that essentially says that if $\int_{-\infty}^{\infty} \frac{\log Q_N(x)}{1+x^2} dx$ is convergent, then the corresponding moment problem is undetermined. An analogous case holds for the Stieltjes case.

From (1.6) we have:

$$\begin{aligned} \log \rho(x) &= \log \omega(x) + \log(p_N(x) p'_{N-1}(x) - p'_N(x) p_{N-1}(x)) \\ &= \log \omega(x) + \log Q_N(x) \,, \end{aligned} \tag{4.1}$$

with $Q_N(x)$ a polynomial. We write the expression in the second way, to emphasize that we just have the addition of a polynomial term. Then:

$$\int_{-\infty}^{\infty} \frac{\log Q_N(x)}{1+x^2} dx < \infty \tag{4.2}$$

and

$$\int_{0}^{\infty} \frac{\log Q_N(x)}{\sqrt{x}(1+x)} dx < \infty \,. \tag{4.3}$$

Therefore, the convergence or divergence of $\int_{-\infty}^{\infty} \frac{\log \rho(x)}{1+x^2}\, dx$ and $\int_{0}^{\infty} \frac{\log \rho(x)}{\sqrt{x}(1+x)} dx$ is exclusively given by the convergence or divergence of $\int_{-\infty}^{\infty} \frac{\log \omega(x)}{1+x^2}\, dx$ and $\int_{0}^{\infty} \frac{\log \omega(x)}{\sqrt{x}(1+x)}\, dx$, respectively. Then, taking into account Krein proposition, it turns out that the matrix models characterized by an indeterminate weight function, have different (but indeterminate) density of states but the same partition function. Thus, we have seen that the density of states is an indeterminate function. Let us show explicitly that their moments are all equal:

$$\rho_\theta(x) = \omega_\theta(x) Q_N(x) \,, \tag{4.4}$$

that is clearly θ-dependent and the moments:

$$\delta_{n,\theta} = \int x^n \omega_\theta(x) Q_N(x) dx = \delta_n \tag{4.5}$$

are θ-independent, since the polynomial $Q_N(x)$ is always the same and then we are led to a sum of moments of the indeterminate function $\omega_\theta(x)$, each one of them θ-independent. Now, we can show that the partition functions are identical. We use the following expression:

$$Z_{N,\theta} = \int \rho_N(x) dx = \int \omega_\theta(x) Q_N(x) dx = Z_N \,. \tag{4.6}$$

That is, the partition function as the 0-moment of the density of states. Alternatively, the usual expression for the partition function in terms of the orthogonal polynomials can be used[8]:

$$Z_N = \frac{N!}{\prod_{i=0}^{N-1} a_i^2}\,, \tag{4.7}$$

where a_i are the following coefficients of the associated orthogonal polynomials: $P_N(x) = a_N x^N + \cdots$. This expression is valid for any weight function and only depends on the associated orthogonal polynomials, that are the same for any θ.

Then, for example, all matrix models with the weight functions (3.3), have the same orthogonal polynomials (essentially the Stieltjes–Wigert,[20] see above). Nevertheless, it is obvious that each will have a different density of states, since we have a different weight function (as we have seen in the previous figures).

In any case, it is clear that approaching matrix models only from the perspective of orthogonal polynomials, there is no *a priori* reason to choose the log-normal weight, since any member of the infinite family has identical orthogonal polynomials. Furthermore, the family of functions discussed above is by no means the only one. From our analysis with the Mellin transform, it is clear that many other functions instead of sin, can lead to the same phenomena. In addition, examples such as[3]:

$$\omega(x) = \frac{x^{-5/2}}{(-x;q)_\infty(-q/x;q)_\infty}\,, \qquad 0 < x < \infty\,, \tag{4.8}$$

can be relatively easily found in the literature. This weight function also leads to the Stieltjes–Wigert polynomials, that are actually terminating basic hypergeometric series and have intimate relationships with theta functions.[3] In this work, we have discovered their remarkable connection with Chern–Simons theory and quantum topological invariants. In general, it turns out that the set V of solutions to an indeterminate moment problem contains very different types of measures: measures $\mu \in V$ with a C^∞-density, discrete measures and measures which are continuous singular.[4] Explicit cases of these last types are found in Ref. 6.

5. Conclusions and Outlook

We have seen that the matrix models that appear in Chern–Simons theory are exactly solvable and that have many interesting properties, which are novel in comparison with the matrix models that appear in 2D quantum gravity. Therefore, the orthogonal polynomials technique constitute an interesting nonperturbative method. We have checked this for the simple case of S^3 and gauge group $\mathrm{U}(N)$ and, interestingly enough, the use of q-deformed orthogonal polynomials readily leads to the usual parameter of Chern–Simons theory. It seems feasible, and certainly an interesting task, to try to extend this result to other geometries, like lens spaces, and to other gauge groups as well.

On the other hand, the non-uniqueness property that we have established does not only seem to be interesting from a mathematical perspective, but it may have some physical meaning as well. We have emphasized the fact that the original Chern–Simons matrix model can be deformed in infinitely many ways, leaving the partition function always unmodified. Since this is a Chern–Simons partition function, we know it is a topological invariant. A well-known result in topology says that the topological invariant of a three-manifold is independent of the triangulation employed to compute the invariant. Thus, one is tempted to try to interpret all these matrix models as representing different triangulations of the manifold. Thus, in a sense, it would be interesting to try to obtain an analogous interpretation to the role of more ordinary matrix models in 2D quantum gravity, but in three dimensions. This is a bit speculative line of research, but it may still be worthwhile. In any case, the non-uniqueness properties are a distinctive mathematical feature of the models and the way that it appears seems also rather interesting in itself, since the resulting logarithmic oscillations are quite typical of physical models where a quantum group structure is present. A more detailed account of this will appear elsewhere.

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