

QUANTUM GROUP SYMMETRY AND DISCRETE SCALE INVARIANCE: SPECTRAL ASPECTS

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ABSTRACT. We study analytical aspects of a generic q -deformation with q real, by relating it with discrete scale invariance. We show how models of conformal quantum mechanics, in the strong coupling regime and after regularization, are also discrete scale invariant. We discuss the consequences of their distinctive spectra, characterized by functional behavior. The role of log-periodic behavior and q -periodic functions is examined, and we show how q -deformed zeta functions, characterized by complex poles, appear. As an application, we discuss one-loop effects in discretely self-similar space-times.

1. INTRODUCTION

The number of physical models that lead to a quantum mechanical spectrum with functional behavior is remarkable [1]–[6]. However, this is not a very well-known fact and indeed, most of the works that possess at least this common feature appear rather scattered in the literature. By functional behavior we mainly refer here to spectra with exponential (either exactly or exponential in the semiclassical region) growth. The usual models of ordinary quantum mechanics certainly seem to imply that only polynomial or algebraic behavior in the quantum numbers can be obtained.

This type of spectral behavior goes hand in hand with the presence of a quantum group symmetry. Indeed, in ordinary quantum mechanics, when q -deformed, the usual polynomial behavior of the spectrum jumps into an exponential-like behavior for the eigenvalues [7, 8]. Is possible to obtain this type of spectra within ordinary quantum mechanics? The answer turns out to be positive, but for a rather particular type of models. More precisely, models of conformal quantum mechanics [9]. These are models that are characterized by a singular potential [10], like $V(r) = -\frac{\lambda}{r^2}$, for example. These models have been notoriously revisited in recent years [1, 2]. Their interest partially relies on the fact that they are a good laboratory to test many quantum field theory features (like regularization and renormalization), but also due to its physical relevance in many different areas, as we shall see. Qualitatively, one may argue that the dimensionless ¹ parameter λ , is willing to play the role of the dimensionless q (rather, $\log q$) parameter. Nevertheless, this is not possible since we do not have an energy scale, given by a more conventional dimensionful parameter. This parameter is given by the usual and necessary cutoff employed in the regularization (or appearing by dimensional transmutation [2] for example). Then, this new parameter behaves like a usual

¹Due to the concomitant symmetry of the potential, of degree -2 , with the kinetic energy term.

parameter in ordinary quantum mechanics and then λ is legitimated to act as an *honest* q parameter. At any rate, the resulting spectrum:

$$(1.1) \quad E_n = E_0 e^{-n\mu},$$

with E_0 depending on the cutoff and μ depending on λ is rather conclusive and certainly close to some typical q -deformed spectra [3].

A feature of q deformed models that we want to stress is their relationship with the concept of discrete-scale invariance [11]. Discrete scale invariance (DSI) is a symmetry that is weaker than the well-known continuous scale invariance. In discrete scale invariance, we have scale invariance under a generic transformation $x \rightarrow \lambda x$ only for specific values of the parameter λ . In general, these values form an infinite but countable set that can be expressed as $\lambda_n = \lambda^n$, with λ playing the role of a fundamental scaling ratio. Regarding q -deformed models, is not difficult to realize that they are DSI invariant with the parameter q playing the role of λ . In addition, since continuous scale invariance is equivalent to continuous translational invariance expressed on the logarithms of the variables [11], then DSI can also be considered as translational invariance, but restricted to a discrete set. That is, only valid for translations of the fundamental unit $\log \lambda$. This is essentially the well-known *lattice* structure associated to q -deformations (see [12] for example).

Already in the seventies, in nuclear physics work -more precisely, the Efimov effect [13]-, we find evidence of the physical relevance of a quantum mechanical spectrum with an exponential behavior. Also in recent work [5], there is emphasis in a spectrum of the type (1.1), and also its relationship with limit cycles in renormalization group flows [14]-[17] and under which conditions the spectrum is of pure geometric growth. Needless to say, discrete scale invariance might lead to any kind of functional behavior for the spectrum and another, closely related, possibility is:

$$(1.2) \quad E_n = \frac{E_0}{\sinh(n\alpha)},$$

that also appears, for example, in [6], and is probably somewhat closer to q -deformed models than (1.1). As we shall see, this spectrum, with linear growth near the ground-state and exponential one for high-lying eigenvalues, presents some interesting features when compared to (1.1).

The paper is organized as follows. In the next Section we discuss how quantum mechanical models with conformal symmetry posses, after regularization, common properties with q -deformed models. Then, in Section 3, we study some spectral functions -like the partition function or the density of states- associated with a purely exponential spectrum. We discuss the role of the complex poles and when these poles induce fractal behavior. Then, we proceed identically, but for a typical q -deformed spectrum. For this, we employ some mathematical works on q -deformed zeta functions, and we see that there is a considerably richer oscillatory behavior, due to an interesting meromorphic structure of the Mellin transforms. In the last Section, we study one-loop effects in discretely self-similar space-times, that both shows a physical application of q -deformed zeta functions and also exploits the connections between quantum group symmetries and discrete scale invariance. In the Conclusions, we present a brief summary and some avenues for future research are suggested.

2. REGULARIZATION IN CONFORMAL QUANTUM MECHANICS AND QUANTUM GROUP SYMMETRIES

As we have mentioned in the Introduction, quantum mechanical models with conformal symmetry, characterized by singular potentials, possess several rather special features. We shall mainly exemplify our discussion with the potential $V(r) = -\frac{\lambda}{r^2}$. As already mentioned, the parameter λ is, in contrast to ordinary quantum mechanics, adimensional, due to the concomitant symmetry with the kinetic term. Note that this leads to conformal symmetry. Actually, it is imposed by it.

In this model, big values of the parameter λ are known to lead to the so-called strong coupling regime where regularization and renormalization are mandatory. This may be done in several ways, as has been discussed in detail [1, 2]. In any case, either by the presence of a dimensional cutoff or by dimensional transmutation for example, one finally ends up with an additional dimensional parameter. This parameter gives an energy scale and the result is that it allows. There are many features of the model that put into evidence the discrete scale invariance. Note that in q -deformations one typically begins with ordinary quantum mechanics, with a dimensional parameter (like ω in a harmonic oscillator), and then the generalization of the algebra is achieved with the introduction of an adimensional parameter q . Roughly speaking, the final situation is the same but the steps are done in opposite order.

However, it must be stressed that while the introduction of a regularization parameter leads to an exponential spectrum (1.1), a posterior renormalization [18] eliminates the excited states and leaves only the ground state. However, since regularization of the singular and, eventually, non-physical part of the potential is a meaningful procedure in many physical applications and in any case, this type of spectra explicitly appears in physical problems, we shall be studying some of its consequences and how they lead to log-periodic behavior, the signature of discrete scale invariance.

Note also that the *intriguing* [2] oscillatory behavior associated to this model in the strong-coupling regime, as exemplified through its wavefunction is again easily understood and even expected in terms of discrete scale invariance. For $r > a$ and energy $E < 0$ [2, 10],

$$(2.1) \quad v^{(>)}(r) = \begin{aligned} & A_{l,\nu} K_{i\Theta}(\kappa r) \\ & \stackrel{(a < r \rightarrow 0)}{=} -A_{l,\nu} \sqrt{\frac{\pi}{\Theta \sinh(\pi\Theta)}} \\ & \times \sin \left\{ \Theta \left[\ln \left(\frac{\kappa r}{2} \right) + \gamma \right] \right\} [1 + O([\kappa r]^2)] , \end{aligned}$$

where $K_{i\Theta}(z)$ is the Macdonald function of imaginary order $i\Theta$, γ is the Euler-Mascheroni constant, and

$$(2.3) \quad \Theta = \sqrt{\lambda - (l + \nu)^2} ,$$

while $\nu = d/2 - 1$, $\lambda = 2mg/\hbar^2$, and $\kappa^2 = -2mE/\hbar^2$. In (2.1)-(2.2) and thereafter, the reduced function $v(r)$ in d dimensions is defined in terms of the separable solution $\Psi(\mathbf{r}) \propto Y_{lm}(\mathbf{\Omega}) v(r)/r^\nu$, with $Y_{lm}(\mathbf{\Omega})$ being the hyperspherical harmonics.

In the strong coupling regime for $\lambda \geq (l + \nu)^2$, (2.2) displays, near the origin, log-periodic oscillatory behavior. We would like to emphasize that while this is usually considered an *intriguing* property [2], this type of behavior turns out to be the

signature of discrete scale-invariance and, as we shall see in the next sections, is ever present in many physical quantities in this and related models. Again, in a comparison with q -deformed models (with q real), notice that with the natural definition of a q -parameter $q = e^{\frac{2\pi}{\Theta}}$, the wavefunction satisfies:

$$(2.4) \quad v^{(>)}(qr) = v^{(>)}(r),$$

this q -periodic property, is actually rather ubiquitous in the context of q -deformations and q -calculus, as we shall see. In addition, it appears when studying other features associated to this model, like the presence of a limit cycle in renormalization group flows. Note that (2.4) can also be considered as a particular case of self-similarity, and thus the connection with fractal behavior can be hinted as well.

3. SPECTRAL BEHAVIOR: LOG-PERIODIC OSCILLATIONS

In this Section, we shall study the statistical mechanics quantities associated to spectra such as (1.1) and (1.2). Namely, partition functions and density of states. We shall study these spectral functions employing the well-known tool of zeta functions [19] (that is to say: an asymptotic study using Mellin transforms [20]). In this sense, we expect the approach to have some interest from a methodological point of view. In particular, it shows how easily the usual framework of the zeta functions employed in physics -mainly known as zeta regularization or heat kernel techniques [19, 21] - has to be enhanced already with relatively simple quantum mechanical models. Typically, the usual zeta functions are characterized by a rather constrained meromorphic structure. In contrast, the models here discussed easily develop an infinite number of complex poles in the Mellin transform of the spectral function. As we shall see, these complex poles are intimately related to fractal behavior [22, 11]. Recall now that the usual framework employed in heat-kernel approaches is borrowed from the theory of pseudodifferential operators and Riemannian geometry. More precisely, in the theory of pseudodifferential operators (Ψ DO) the relation between the heat kernel and zeta functions is the following. Let A a pseudodifferential operator (Ψ DO), fulfilling the conditions of existence of a heat kernel and a zeta function (see, e.g., [19]). Its corresponding heat kernel is given by (see [19], and references therein):

$$(3.1) \quad K_A(t) = \text{Tr} e^{-tA} = \sum_{\lambda \in \text{Spec } A} ' e^{-t\lambda},$$

which converges for $t > 0$, and where the prime means that the kernel of the operator has been projected out before computing the trace, and once again the corresponding zeta function:

$$(3.2) \quad \zeta_A(s) = \frac{1}{\Gamma(s)} \text{Tr} \int_0^\infty t^{s-1} e^{-tA} dt.$$

For $t \downarrow 0$, we have the following asymptotic expansion:

$$(3.3) \quad K_A(t) \sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0,$$

being:

$$\begin{aligned}
 (3.4)_n(A) &= \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \operatorname{Res}_{s=s_j} \zeta_A(s), \quad \text{if } s_j \notin Z \text{ or } s_j > 0, \\
 \alpha_j(A) &= \frac{(-1)^k}{k!} \left[\operatorname{PP} \zeta_A(-k) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \gamma \right) \operatorname{Res}_{s=-k} \zeta_A(s) \right], \\
 \beta_k(A) &= \frac{(-1)^{k+1}}{k!} \operatorname{Res}_{s=-k} \zeta_A(s),
 \end{aligned}$$

Zeta functions with complex poles have already appeared in the literature. First, the problem of the study of vibrations in the presence of irregular boundaries or shapes has already been undertaken as a generalization of spectral problems with smooth geometries [22]. On the other hand, in the study of combinatorial structures [20], Mellin transforms with complex poles are the most usual and natural object and, in particular, extremely simple functions in analytic number theory possess this property [20].

Regarding the appearance of poles with imaginary part $s^* = \sigma + it$, we should only mention that they imply the presence of fluctuations of the type $x^{-s^*} = x^{-\sigma} \exp(it \log x)$ in the expansion of the original function. Very often, regularly spaced poles appear, leading to a Fourier series in $\log x$. The following (simplified) table gives an idea of the correspondence between the original function and the singularity structure of its associated Mellin transform [20]:

$f^*(s)$	$f(x)$
Pole at ξ	Term in asymptotic expansion $\approx x^{-\xi}$
Multiple pole: $\frac{1}{(s-\xi)^k}$	Logarithmic factor: $\frac{(-1)^k}{k!} x^{-\xi} (\log x)^k$
Complex pole: $\xi = \sigma + it$	Fluctuations: $x^{-\xi} \exp(it \log x)$

3.1. Pure geometric growth. Without losing generality, let us begin by showing the features of the spectrum $\lambda_n = 2^n$. The associated partition function is:

$$(3.6) \quad K(t) = \sum_{n=0}^{\infty} e^{-2^n t}.$$

We can construct an associated zeta function as:

$$(3.7) \quad \zeta(s) = \sum_{n=0}^{\infty} 2^{-sn} = \frac{1}{1 - 2^{-s}},$$

it is a geometric series and has infinitely many (complex) poles:

$$(3.8) \quad s_k = \frac{2i\pi k}{\log 2}, \quad k = 0, \pm 1, \pm 2, \dots$$

with residues $1/\log 2$. Since the corresponding Mellin transform is $\Gamma(s)\zeta(s)$, we have a double pole at 0, the infinite sequence of complex poles and the poles of the Gamma function at the negative integers. Then, the asymptotic expansion (see Appendix) is:

$$(3.9) \quad K(t)_{t \rightarrow 0} \sim -\log_2(t) - \frac{\gamma}{\log 2} + \frac{1}{2} - Q(\log_2 x) + \sum_{n=0}^{\infty} \frac{1}{1 - 2^n} \frac{(-t)^n}{n!}.$$

Where $Q(\log_2 x)$ is the contribution from the imaginary poles:

$$(3.10) \quad Q(\log_2 x) = \frac{1}{\log 2} \sum_{k \neq 0} \Gamma(s_k) \exp(-2ik\pi \log_2 x),$$

which is a fluctuating term of order $O(1)$. Then, we have found that the $t \rightarrow 0$ expansion for this system contains a term that is a Fourier series in $\log_2 x$, with coefficients of Gamma type. Note that these oscillations are logarithmic oscillations, typical in discrete-scale invariant models [11]. Notice that $|\Gamma(\pm 2\pi i / \log 2)| = 0.545 \cdot 10^{-6}$, and, in addition, $\Gamma(s)$ has a strong decay while progressing through the imaginary line. More precisely, recall the complex version of Stirling's formula [23]:

$$(3.11) \quad |\Gamma(\sigma + it)| \sim \sqrt{2\pi} |t|^{\sigma-1/2} e^{-\pi|t|/2}.$$

Thus, the terms coming from higher poles in the imaginary axes are strongly damped. Then, the attentive reader may profitably wonder whether the fluctuations in our models imply that the resulting function is fractal². Actually, regarding $K(t)$ it turns out that the great damping in the amplitudes due to the Gamma function, as explained above, prevents this to be the case [22]. Therefore, we are dealing with small fluctuations, with tiny *wobbles* [20].

This is not the case of the density of states:

$$(3.12) \quad \rho(x) = \sum_{n=0}^{\infty} \delta(x - \lambda_n),$$

since the Mellin transform of the density of states is given by $\zeta(1-s)$, and then the expansion for the density of states is:

$$(3.13) \quad \rho(x)_{x \rightarrow \infty} \sim \frac{x^{-1}}{\log 2} \sum_{k=-\infty}^{\infty} \cos(2k\pi \log_2 x).$$

Let us explore some other physically interesting spectral function to learn more about the role of the poles and its residues. In some contexts, q deformations have proven useful in the study of disordered systems, where is interesting to understand the conductance, given by:

$$(3.14) \quad \langle g(t) \rangle = \sum_{n=0}^{\infty} \frac{1}{1 + \lambda_n/t},$$

with λ_n have, in principle, a behavior of the type discussed in this Section. Once again, we can set $\lambda_n = 2^n$ without losing generality, and compute the Mellin transform:

$$(3.15) \quad g(s) = \int_0^{\infty} \langle g(t) \rangle t^{s-1} dt = \frac{\pi}{\sin \pi(s-1)} \frac{1}{1-2^s},$$

and taking into account all the poles in the Mellin transform, we obtain:

$$(3.16) \quad \langle g(t) \rangle_{t \rightarrow \infty} \sim \log_2 t + \frac{1}{2} + P(\log_2 t) + \sum_{k=1}^{\infty} \frac{(-1)^k}{1-2^k} t^{-k},$$

²That is, continuous but differentiable nowhere.

where $P(\log_2 t)$ is the periodic function coming from the complex poles $s_k = \frac{2\pi ik}{\log 2}$:

$$(3.17) \quad \begin{aligned} P(\log_2 t) &= \frac{1}{\log 2} \sum_{k \in \mathbb{Z}} \frac{\pi}{\sin \pi(s_k - 1)} \exp(-2ik\pi \log_2 t) \\ &= -\frac{2\pi}{\log 2} \sum_{k=1}^{\infty} \frac{\sin 2\pi k \log_2 t}{\sinh\left(\frac{2\pi^2 k \log_2 t}{\log 2}\right)}. \end{aligned}$$

Once again, the important point is that the function $\frac{\pi}{\sin \pi(s-1)}$ exhibits an exponential decrease along vertical lines. Then, the contribution of the periodic function is truly small. This can be readily appreciated from the sinh factor, that severely damps the contribution of higher poles. More precisely, the function satisfies $|P(t)| \leq 7.8 \cdot 10^{-12}$. Nevertheless, it is plain that they can have a more considerable effect (for example, by decreasing the value of q , that we have set to $\frac{1}{2}$, just to give precise numerical values). In any case, it is interesting to point out that even such minute fluctuations have lead to discrepancies in analytic studies of algorithms [20].

3.2. q -deformed growth. In the previous Section we have studied in detail the case of a pure geometric growth. Let us now pay attention to the very well-known case of a q -deformed spectrum. From a practical point of view, numbers are substituted by q -numbers:

$$(3.18) \quad n \rightarrow [n]_q = \frac{1 - q^n}{1 - q}.$$

Note that the resulting spectrum exhibits exponential growth for the high-level eigenvalues and linear growth for the low-lying eigenvalues:

$$(3.19) \quad \begin{aligned} \lim_{n \rightarrow \infty} [n]_q (q - 1) &\sim e^{n \log q} \\ \lim_{n \rightarrow 0} [n]_q (q - 1) &\sim n \log q. \end{aligned}$$

The effect of this spectrum -instead of a purely exponential one- in the meromorphic structure of the associated zeta function is not evident a priori.

Interestingly enough, it turns out that there already exist some few interesting works dealing with q deformed zeta functions [24, 26, 25]. Consider, for example, [26]:

$$(3.20) \quad \zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^{sn}}{[n]_q^s}.$$

The meromorphic continuation to all s can be easily obtained by application of the binomial expansion $(1 - q^n)^{-s} = \sum_{r=0}^{\infty} \binom{s+r-1}{r} q^{nr}$ [24]:

$$(3.21) \quad \zeta_q(s) = (1 - q)^s \sum_{r=0}^{\infty} \binom{s+r-1}{r} \frac{q^{s-1+r}}{1 - q^{s-1+r}}.$$

Thus, the set of poles is $s_{k,n} = -k + \frac{2\pi in}{\log q}$ with $k = 0, -1, -2, \dots$ and n an integer. Hence, as expected, we are lead to complex poles and thus to logarithmic oscillations and, eventually, to fractal behavior as discussed previously. Nevertheless, in comparison with the previous Section, the singularity pattern is manifestly much

richer. Since there are many possible q deformations, it is worth to introduce, following [25], a generic two-variable zeta function:

$$(3.22) \quad f_q(s, t) = \sum_{n=1}^{\infty} \frac{q^{nt}}{[n]_q^s} \quad \text{with } (s, t) \in \mathbb{C}^2,$$

and proceed identically. Namely, binomial expansion and change of the order of summation [25]:

$$(3.23) \quad f_q(s, t) = (1-q)^s \sum_{r=0}^{\infty} \binom{s+r-1}{r} \frac{q^{t+r}}{1-q^{t+r}}.$$

In [25], it is argued that $t = s - 1$ provides the "right" q -deformation of Riemann's zeta function, instead of the case $t = s$ presented above and discussed in [24]. Certainly, we are led to:

$$(3.24) \quad f_q(s, s-1) = (1-q)^s \left(\frac{q^{s-1}}{1-q^{s-1}} + s \frac{q^s}{1-q^s} + \frac{s(s+1)}{2} \frac{q^{s+1}}{1-q^{s+1}} + \dots \right),$$

and consequently the poles are simple at $1 + \frac{2\pi in}{\log q}$ with $n \in \mathbb{Z}$ and $j + \frac{2\pi ik}{\log q}$ with $j, k \in \mathbb{Z}$ and $j \leq 0, k \neq 0$.

Therefore, in comparison with (3.20), we avoid the poles at negative integer values and at the origin. In addition, the corresponding values at these points approach the known ones, when $q \rightarrow 1$. Furthermore, the convergence to $\zeta(s)$ when $q \rightarrow 1$ for any s is obtained. This is also possible for (3.20), but up to some terms in q [24]. With the previous information, it is straightforward now to study the asymptotic behavior of the associated partition function:

$$(3.25) \quad K(t) = \sum_{n=1}^{\infty} q^{-n} e^{-\frac{q^n}{[n]_q} t},$$

$$(3.26) \quad K(t)_{t \rightarrow 0} \sim \frac{(1-q)}{t} \sum_{k=-\infty}^{\infty} \Gamma\left(1 + \frac{2\pi ik}{\log q}\right) \cos\left(2\pi k \frac{\log x}{\log q}\right) \\ + \sum_{n=0}^{\infty} \zeta_q(-n) \frac{(-t)^n}{n!} \\ + \sum_{j=0}^{\infty} \sum_{k \neq 0} t^j \Gamma(s_{j,k}) (s_{j,k})_j (1-q)^{s_{j,k}} \cos\left(2\pi k \frac{\log x}{\log q}\right).$$

As mentioned, the zeta values $\zeta_q(-n)$ are finite and its value well-known [25]. The $s_{j,k}$ denote the poles and $(s_{j,k})_j$ the Pochhammer symbol (also known as rising factorial) [23].

However, as explained in the Introduction, we are interested in spectrum of the type $\lambda_n = \sinh(n \log q)$. This directs our interest to (3.23) with $t = s/2$. Thus:

$$(3.27) \quad \zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^{ns/2}}{[n]_q^s} = \left(\frac{q-1}{2}\right)^s \sum_{n=1}^{\infty} \sinh^{-s}\left(\frac{n \log q}{2}\right) \\ = (1-q)^s \sum_{r=0}^{\infty} \binom{s+r-1}{r} \frac{q^{s/2+r}}{1-q^{s/2+r}}.$$

This zeta function also possesses many interesting theoretical properties. In particular, the mathematical relevance -in the theory of special functions and in harmonic analysis- of the transformation $x^s \rightarrow \sinh^s x$ is discussed in [26]. Additionally, it actually appears in a natural way in physical applications. In random matrix models for example, a model with the usual Gaussian potential but with $(x_i - x_j)^2 \rightarrow \sinh^2((x_i - x_j)/2)$ as a correlation factor, appears in Chern-Simons theory [27, 28].

The set of poles is $2a + \frac{2\pi ib}{\log q}$ with $a, b \in \mathbb{Z}$ and $a \leq 0$. As usual, the set of poles allows to obtain the rich asymptotic behavior of the density of states or the partition function. In any case, note that the contribution of the additional poles is subleading against the rightmost line of complex poles (and in comparison with the purely geometric growth case).

4. QFT ON A DISCRETELY SELF-SIMILAR SPACETIME: ONE-LOOP EFFECTS.

Now, we shall show how q -deformed zeta functions (and the corresponding heat kernels) and the above mentioned q -periodic functions, naturally appear in essentially the same way as ordinary zeta functions do in zeta regularization/heat-kernel studies [19, 21]. Namely, in the study of one-loop effects on a curved space-time background.

For this, we focus our attention on a certain type of space-times that appear in the study of critical phenomena in gravitational collapse. Indeed, from Choptuik's groundbreaking work [29], much attention has been recently devoted to metrics with a discrete self-similarity [30, 31]. A spacetime is discretely self-similar if there exists a discrete diffeomorphism Φ and a real constant Δ such that:

$$(4.1) \quad \Phi^* g_{ab} = e^{-2\Delta} g_{ab},$$

where $\Phi^* g_{ab}$ is the pull-back of g_{ab} under the diffeomorphism Φ .

Notice that, in contrast with the continuous case, this definition does not introduce an homothetic vector field ξ . The parameter is essentially the analogous of $\log q$ in our discussion. In Schwarzschild-like coordinates, the spacetime line element reads:

$$(4.2) \quad ds^2 = -\alpha^2(r, t) dt^2 + a^2(r, t) dr^2 + r^2 d\Omega^2,$$

where the coefficients satisfy the property:

$$(4.3) \quad \alpha(e^{\Delta} r, e^{\Delta} t) = \alpha(r, t) \quad \text{and} \quad a(e^{\Delta} r, e^{\Delta} t) = a(r, t),$$

the by-now familiar q -periodic property. In spite of the interest they have generated, these metrics are still much less studied than the usual continuously self-similar spacetimes, for example. Let us also briefly discuss the q -periodic property (4.3).

4.1. On q -periodic functions. Recall that when we deal with a noncommutativity of the type $xy = qyx$ (Manin's *quantum plane*), then ordinary derivatives are substituted by q -derivatives [8, 32]:

$$(4.4) \quad \partial_x^{(q)} f(x; y; \dots) \equiv \frac{f(qx; y; \dots) - f(x; y; \dots)}{(q-1)x}.$$

Notice how the q derivative measures the rate of change with respect to a dilatation of the argument, instead of the translation of the usual derivative. Then, from the previous expression, it is manifest that a q -periodic function satisfies:

$$(4.5) \quad \partial_L^{(q)} g(L) = 0$$

Actually, the unique solution to this equation is a q -periodic function and, of course, as in the classical case, a constant. Therefore, it can be said that a q -periodic function plays, at the level of q -calculus, the role of a constant in ordinary (commutative) calculus.

The connection with complex dimensions [22, 11, 20] can be easily obtained. We consider the Mellin transform of the q periodic function:

$$(4.6) \quad h(s) \equiv \int_0^\infty g(x) x^s dx.$$

Taking into account the following property of Mellin transforms (see [20] for example):

$$(4.7) \quad \int_0^\infty g(qx) x^s dx = q^{-s} h(s),$$

and considering the q -periodic property (4.11):

$$(4.8) \quad q^{-s} h(s) = h(s) \Rightarrow s_k = \frac{2\pi i n}{\log q}, \quad n \in \mathbb{Z}.$$

Thus, the Mellin transform of a q periodic function contains infinitely many complex poles. As we already know, this implies that the q periodic function may be fractal. As we have already seen in the other sections, this depends on the precise form of the function itself. Namely, on the residue corresponding to the poles.

Equivalently, one may argue that any log-periodic term satisfy the restriction imposed by (4.11), so one can construct any suitable combination, such as:

$$(4.9) \quad f(x) = \sum_{n=1}^k a_n \sin\left(2\pi b_n \frac{\log x}{\log q}\right),$$

with rather generic coefficients a_n and b_n . Nevertheless, we are still limiting the finest scale possible, since the sum stops at $n = k$ and thus the function is not a genuine fractal, albeit the oscillatory pattern is certainly much richer than a single log-periodic term. But we can consider a full Fourier series and take $k \rightarrow \infty$. Consider for example, $a_n = n^{-\gamma}$ with $\gamma > 1$; then:

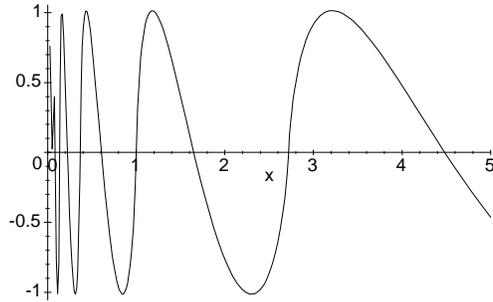
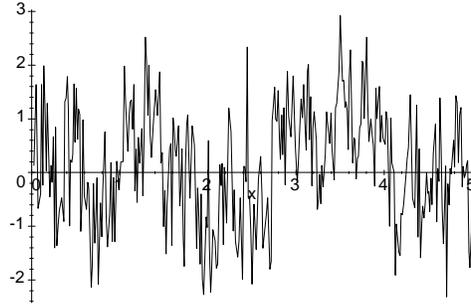
$$(4.10) \quad f(x) = \sum_{n=1}^\infty \frac{1}{n^\gamma} \sin\left(2\pi n \frac{\log x}{\log q}\right),$$

It is well-known that low enough values of γ lead to fractal behavior. Therefore, both relatively simple log-periodic patterns that lead to smooth functions and much more fluctuating patterns leading to fractal functions (see Figures below) are q -periodic.

Another interesting aspect in the study of models characterized by a discrete scale invariance, is the appearance of a limit cycle in a renormalization group analysis. A limit cycle is a 1-parameter family of coupling constants that is closed under the RG flow. The necessary adimensional parameter corresponds to the angle $0 < \theta < 2\pi$ and the scale invariance is discrete. In the several models where a cyclic RG behavior has been found, it turns out that the couplings return to their initial values after a finite RG time λ :

$$(4.11) \quad g(e^\lambda L) = g(L)$$

where L is the RG length scale. Note that many other interesting physical quantities are also characterized by this q -periodic property [6].


 FIGURE 1. $a_n = \frac{1}{n^2}$, $k = 200$ and $b_n = n$

 FIGURE 2. $a_n = \frac{1}{n^{0.3}}$, $k = 200$ and $b_n = n^2$

Needless to say, from a physical standpoint one would like, perhaps, to rule out the fractal case, since this implies the non-differentiability of $g(L)$. Further work to clarify this would be desirable, especially taking into account that one can construct intermediate cases, where the resulting function is neither smooth nor fractal, but begins to develop a countable set of singularities [33].

4.2. Discretely self-similar space-time as a quantum group. As we have just seen, discrete selfsimilarity leads to coefficients of the metric which are the q -periodic functions previously discussed. So, already at the classical level, the presence of the discrete self-similarity may imply a very low differentiability of the space-time. This may not necessarily jeopardize the validity of such a spacetime [34].

This happens already at the level of the classical geometry, but what about one-loop effects? Let us look at the heat kernel. To begin with, one can not directly apply the typical methods [19, 21]. Consider, the symbol of a Laplacian operator on a Riemannian manifold:

$$(4.12) \quad a(x, \xi) = -g^{ij}(x)\xi_i\xi_j + B^i(x)\xi_i + C(x).$$

Then, as in [35], a signal that the usual framework can not be applied is that the metric (4.2)-(4.3) leads to an oscillating symbol. The difference with [35] is that in the present case the oscillatory behavior is log-periodic.

Now, it is manifest that the existence of the Δ parameter, the equivalent of $\log q$, will lead to functional behavior in the spectra of the Laplacian. For definiteness, let us briefly consider a very simple particular example. Consider the one-dimensional case:

$$(4.13) \quad ds^2 = g(x)dx^2,$$

where $g(x)$ is of course q -periodic. Then, solving for the spectra of the Laplacian implies:

$$(4.14) \quad g(x)\phi''(x) = E_n\phi(x),$$

And the q -periodic property of $g(x)$ readily implies that $\phi(qx) = q^{-2}\phi(x)$. Thus, the self-similarity of the wavefunction is an automatic consequence. Note that this already implies self-similarity of the full heat-kernel $K(x, y, t, D) = \langle x | \exp(-tD) | y \rangle$. Of course, such a wavefunction can now be expanded in a log-periodic Fourier series. However, it is simpler to notice that a function satisfying $\phi(qx) = q^{-2}\phi(x)$ can also be given by:

$$(4.15) \quad \phi(x) = \sum_{n=-\infty}^{\infty} (1 - \exp(iq^n x)) q^{2n}$$

Although we have written the complex wavefunction, it is clear that these are usual trigonometric oscillations with the only novelty that the frequencies of the oscillations are of the type $\omega_n = q^n$. That is to say, with the geometric growth we have been discussing in this paper. This obviously translates into a geometric growth for the energy eigenvalues and thus, to a heat kernel and zeta functions of the type discussed in the previous Section.

Notice that all the previous discussions on the appearance of fractal behavior are now relevant as we already know that this novel behavior in the short-time asymptotics of the trace of the heat-kernel, is of very small amplitude. Recall that this expansion gives the large mass expansion of the effective action [21].

Therefore, we have seen that, at a quantum level, discretely self-similar space-times are very much related to quantum groups³. It seems appealing that the study of critical phenomena in gravitational collapse, that naturally leads to the discrete symmetry characterized by Δ , might be related with quantum group symmetries. This is so because a quantum group symmetry is expected to be a key ingredient in quantum gravity [36] and, consequently, in black hole physics. Thus, it seems appealing that Choptuik's result could be interpreted as a signature of a quantum group symmetry and thus, a full-fledged quantum gravity effect.

5. CONCLUSIONS AND OUTLOOK

We have focussed on analytical aspects of q -deformations with q real. This has been done by simple comparison with discrete scale invariance [11], a well-known topic in many statistical physics applications (although often appearing in a scattered fashion in the literature, in spite of reviews such as [11]). We have seen

³Recall that we can consider a quantum group as a manifold, and its spectral analysis is of the type here discussed [24].

that some models of conformal quantum mechanics, in the strong coupling regime an after regularization, also lead to the same behavior. In this sense, once this connection is realized, the properties in this regime are not intriguing but rather expected ones.

Incidentally, the tight connections between discrete scale invariance and fractal geometry puts into evidence the relationship between such models, q -deformations and fractal behavior. In addition, the functional behavior in the spectra, which is a consequence of the q -deformation or, more generically, of the presence of a discrete scale symmetry, naturally leads to associated spectral functions (such as zeta functions and trace of heat kernels) with interesting behavior. While such functions have appeared in mathematical literature, they have not been considered in zeta regularization [19, 21] and in this context, they show many interesting novel features. Mainly, a very rich meromorphic structure, which is of course consequence of the discrete scale invariance. This seems to be interesting, but the main reason is not one of just mathematical generality, as we have also shown that quantum field theory on discretely self-similar space-times exactly requires such an extended framework.

Precisely, regarding the discretely self-similar space-times, there seems to be several interesting open questions. On a mathematical level, it seems interesting to further study the low differentiability properties of some of the space-times included in the generic definition (4.2)-(4.3). This may be done along the lines of [34] for example. From a more physical point of view, we hope to have shown that quantum field theory on these backgrounds is an interesting and rather unexplored problem. To begin with, there is the connection with quantum group symmetries and the possible meaning in quantum gravity, perhaps even also the comparison with developments on noncommutative field theory. If one is more interested in technical issues, we have already seen that in a heat-kernel/zeta functions approach, novel behavior appears, so a more careful study of this aspect may be worth as well, maybe in comparison with heat-kernels on noncommutative spaces [35].

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APPENDIX A. MELLIN TRANSFORMS AND ZETA FUNCTIONS: COMPLEX POLES

To complement the information provided in the Introduction, we just quote here the theorem that is implicitly used in the computations of the asymptotic behavior of the spectral functions. We note in passing that not only a geometric growth of the spectral sequence $\{\lambda_n\}$ leads to complex poles. An arithmetic sequence with varying amplitudes (degeneracies) may exhibit the same behavior. That is, Dirichlet series such as:

$$(A.1) \quad L(s) = \sum_n \frac{v_2(n)}{n^{-s}},$$

where $v_2(n)$ denotes the exponent of 2 in the prime number decomposition of the integer n , has complex poles [20]. The theorem is as follows.

Theorem A.1. *Let $f(x)$ be continuous in $]0, \infty[$ with Mellin transform $f^*(s)$ having a non-empty fundamental strip $\langle \alpha, \beta \rangle$.*

(i) *Assume that $f^*(s)$ admits a meromorphic continuation to the strip $\langle \gamma, \beta \rangle$ with some $\gamma < \alpha$ with a finite number of poles there, and is analytic on $\Re(s) = \gamma$.*

Assume also that there exists a real number $\eta \in (\alpha, \beta)$ such that

$$(A.2) \quad f^*(s) = O(|s|^{-r}) \text{ with } r > 1,$$

when $|s| \rightarrow \infty$ in $\gamma \leq \Re(s) \leq \eta$. If $f^(s)$ admits the singular expansion for $s \in \langle \gamma, \alpha \rangle$*

$$(A.3) \quad f^*(s) \asymp \sum_{(\xi, k) \in A} d_{\xi, k} \frac{1}{(s - \xi)^k},$$

then an asymptotic expansion of $f(x)$ at 0 is

$$(A.4) \quad f(x) = \sum_{(\xi, k) \in A} d_{\xi, k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-\xi} (\log x)^k \right) + O(x^{-\gamma}).$$

There is also an analogous statement for an asymptotic expansion of $f(x)$ at ∞ . Proofs can be found in [20]. To conclude, we just want to point out that in many cases, $f^*(s)$ is meromorphic in a complete left or right half plane, and then a complete asymptotic expansion for $f^*(s)$ results. Such an expansion can be convergent or divergent. If divergent, the expansion is then only asymptotic. If convergent it may represent the function exactly. In general, we need a fast enough and uniform decrease of $f^*(s)$ along vertical lines. The example treated in Section 2 for example, satisfies this property and then, the representation is exact. More details can be found in [20].

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